Pure Mathematics Qualifying Exam
September 3, 2005

Part I

1. What sort of graph does the equation

$$x^2 - 4xy + 8y^2 - 4yz + z^2 = 9$$

have? You don't need to sketch the graph; a good description in words is enough.

2. Define the open upper half-plane $U = \{\Im z > 0\}$ and the closed upper half-plane $\bar{U} = \{\Im z \geq 0\}$, where $\Im z$ denotes the imaginary part of z. Find a function $h(z)$ defined on $\bar{U}$ that satisfies the following conditions:

   - h is continuous on $\bar{U} \setminus \{-1, 1\}$ and harmonic on U;
   - $h(x) = 1$ for $-1 < x < 1$, while $h(x) = 0$ for $x < -1$ and for $x > 1$;
   - As $|z|$ tends to infinity with $z \in U$, the value $h(z)$ tends to zero.

3. For any continuous function $g : [-\pi, \pi] \to \mathbb{R}$, define

$$\|g\|_p = \left[\int_{-\pi}^{\pi} |g(x)|^p \, dx \right]^{1/p} \quad \text{for any real number } p \geq 1$$

and

$$\hat{g}(n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-int} g(t) \, dt \quad \text{for any integer } n.$$

Let $f : [-\pi, \pi] \to \mathbb{R}$ be a continuously differentiable real-valued function such that $f(-\pi) = f(\pi)$.

   (a) Show that

$$\|f'\|_2^2 = \sum_{n=-\infty}^{\infty} n^2 \big|\hat{f}(n)\big|^2.$$

   (You may quote standard results about Fourier series.)

   (b) Prove that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| \leq \frac{1}{\sqrt{2\pi}} \|f\|_1 + \frac{\pi}{\sqrt{3}} \|f'\|_2.$$

   (You may use without proof the fact that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.)

4. Let $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ be prime integers. Assume that $q_1, \ldots, q_n$ are distinct. Show that $\sqrt[q_1]{p_1} + \cdots + \sqrt[q_n]{p_n}$ is an irrational number.

(continued on back)

5. Suppose that A is a symmetric $n \times n$ positive definite matrix with real entries, that is, $x^T A x > 0$ for every $x \in \mathbf{R}^n \setminus \{0\}$. Verify that there is an $n \times n$ matrix Q with real entries such that $QQ^T = A$.

6. Using the calculus of residues (contour integration), evaluate the integral

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\, dx.$$

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Part II

1. Let $\mathbf{F}$ be the vector field defined by

$$\mathbf{F}(x, y, z) = (x^2 + x)\mathbf{i} - (3xz - y)\mathbf{j} + (4z + 1)\mathbf{k}.$$

   Let S denote the surface of the sphere given by the equation $x^2 + y^2 + z^2 = 4$.

   (a) Calculate the flux of the vector field $\mathbf{F}$ outwards through S.

   (b) Let S_1 denote the part of S that lies above the xy-plane. Calculate the flux of $\mathbf{F}$ upwards through S_1.

2. Suppose the symmetric group S_n acts transitively on a set X. Show that either $|X| \leq 2$ or $|X| \geq n$.

3. Let I be the $m \times m$ identity matrix and J the $m \times m$ matrix with 1 in every entry.

   (a) Prove that $\det(qI + rJ) = q^{m-1}(mr + q)$ for all real numbers q and r.

   (b) Assume that $q > 0$ and $r > 0$. If A is an $m \times n$ matrix with $AA^T = qI + rJ$, show that $m \leq n$.

4. Suppose that the function f is meromorphic for $|z| < 1$, continuous at every point z with $|z| = 1$, and satisfies $|f(z)| = 1$ when $|z| = 1$. Prove that f is a rational function.

5. Show that

$$2^{1-p} \leq \frac{x^p + y^p}{(x + y)^p} \leq 1$$

   for any $x > 0$, $y > 0$, $p \geq 1$.

6. Let R be a commutative ring (with identity) and let I and J be two ideals in R. Suppose that $I + J = R$, that is, every element of R can be written as $i + j$ for some $i \in I$ and $j \in J$.

   (a) Prove that $I \cdot J = I \cap J$.

   (b) Prove that for any $a, b \in R$ there exists a $c \in R$ such that $c \equiv a \pmod{I}$ and $c \equiv b \pmod{J}$. (Here, the notation $x \equiv y \pmod{I}$ means that $x - y \in I$.)