Applied Mathematics Qualifying Exam
September 3, 2005

Part I

1. What sort of graph does the equation

$$x^2 - 4xy + 8y^2 - 4yz + z^2 = 9$$

have? You don't need to sketch the graph; a good description in words is enough.

2. Define the open upper half-plane $U = \{\Im z > 0\}$ and the closed upper half-plane $\bar{U} = \{\Im z \geq 0\}$, where $\Im z$ denotes the imaginary part of z. Find a function $h(z)$ defined on $\bar{U}$ that satisfies the following conditions:

- h is continuous on $\bar{U} \setminus \{-1, 1\}$ and harmonic on U;
- $h(x) = 1$ for $-1 < x < 1$, while $h(x) = 0$ for $x < -1$ and for $x > 1$;
- As $|z|$ tends to infinity with $z \in U$, the value $h(z)$ tends to zero.

3. For any continuous function $g : [-\pi, \pi] \to \mathbb{R}$, define

$$\|g\|_p = \left[\int_{-\pi}^{\pi} |g(x)|^p \, dx \right]^{1/p} \quad \text{for any real number } p \geq 1$$

and

$$\hat{g}(n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-int} g(t) \, dt \quad \text{for any integer } n.$$

Let $f : [-\pi, \pi] \to \mathbb{R}$ be a continuously differentiable real-valued function such that $f(-\pi) = f(\pi)$.

(a) Show that

$$\|f'\|_2^2 = \sum_{n=-\infty}^{\infty} n^2 \left| \hat{f}(n) \right|^2.$$

(You may quote standard results about Fourier series.)

(b) Prove that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| \leq \frac{1}{\sqrt{2\pi}} \|f\|_1 + \frac{\pi}{\sqrt{3}} \|f'\|_2.$$

(You may use without proof the fact that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.)

(continued on back)

4. The motion $x = x(t)$ of a particle in a symmetric double-well potential is modeled by the following ODE in dimensionless form:

$$x'' + \omega^2(x - x^3) = 0.$$

Here $\omega > 0$ is constant.

(a) Find and classify the type of each equilibrium point.

(b) Plot the phase-plane x' versus x for this conservative system.

(c) Let the initial conditions be $x(0) = 0$ and $x'(0) = x_0$. For what values of x_0 does a periodic solution exist?

5. Define $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(a) Show that A is similar to B.

(b) Verify that

$$e^{tB} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}.$$

(c) Solve the system of differential equations

$$\frac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t), \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

6. Using the calculus of residues (contour integration), evaluate the integral

$$\int_0^\infty \frac{x^{1/3}}{1 + x^2}\, dx.$$

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Part II

1. Let $\mathbf{F}$ be the vector field defined by

$$\mathbf{F}(x, y, z) = (x^2 + x)\mathbf{i} - (3xz - y)\mathbf{j} + (4z + 1)\mathbf{k}.$$

Let S denote the surface of the sphere given by the equation $x^2 + y^2 + z^2 = 4$.

   (a) Calculate the flux of the vector field $\mathbf{F}$ outwards through S.

   (b) Let S_1 denote the part of S that lies above the xy-plane. Calculate the flux of $\mathbf{F}$ upwards through S_1.

2. Let $a > 0$, D, α, and $u_1 > u_0$ be constants. Suppose that a ball of radius $a > 0$ is initially heated to a uniform temperature u_1 and then proceeds to cool off due to Newtonian cooling on the boundary sphere. Assume that the temperature $u(r, t)$ is radially symmetric. An appropriate model is a function u satisfying:

   - $u_t = D(u_{rr} + \frac{2}{r}u_r)$ for $0 \le r \le a$ and $t \ge 0$;
   - $-Du_r = \alpha(u - u_0)$ on $r = a$;
   - u is bounded as $r \to 0$;
   - $u(r, 0) = u_1$ for $0 \le r \le a$.

   (a) What are the physical dimensions of D and α? What is the steady-state solution? (The tranformation $u = v/r$ is helpful here.)

   (b) Show that the eigenvalue relation has the form $\tan z = z/(1-\beta)$ for some constant β. Calculate β explicitly, show that β is dimensionless, and graph the eigenvalue relation. Express the solution as an eigenfunction expansion.

3. Let I be the $m \times m$ identity matrix and J the $m \times m$ matrix with 1 in every entry.

   (a) Prove that $\det(qI + rJ) = q^{m-1}(mr + q)$ for all real numbers q and r.

   (b) Assume that $q > 0$ and $r > 0$. If A is an $m \times n$ matrix with $AA^T = qI + rJ$, show that $m \le n$.

(continued on back)

4. Let $a_1, \ldots, a_n \in \mathbb{R}$. Show that for an appropriate choice of branch, the transformation

$$z \mapsto \left(\prod_{i=1}^{n} (z - a_i) \right)^{1/n}$$

maps the upper half plane into itself. Describe the image of the real axis.

5. Show that

$$2^{1-p} \leq \frac{x^p + y^p}{(x+y)^p} \leq 1$$

for any $x > 0$, $y > 0$, $p \geq 1$.

6. The displacements $y_1(t)$ and $y_2(t)$ for a coupled mass-spring system subject to an external forcing $f(t)$ satisfy the ODE system

$$\begin{aligned} m_1 y_1'' &= -k_1 y_1 + k_2(y_2 - y_1) \\ m_2 y_2'' &= -k_2(y_2 - y_1) + f(t)\,. \end{aligned}$$

   (a) Write this system in the form $y'' = Ay + g$ for some matrix A and some vectors y and g.

   (b) If $m_1 = m_2 = 1$, $k_1 = 5$, $k_2 = 6$, and $f(t) = 0$, find the general solution to this system by first looking for a solution of the form $y = ve^{i\omega_0 t}$ for some unknown vector v and frequency ω_0. (Here $i = \sqrt{-1}$.)

   (c) Now let $f(t) = \sin(\omega t)$. Find a particular solution for this system of the form $y(t) = r\sin(\omega t)$, where r is a vector independent of t but dependent on ω that is to be found. Give a rough plot of $|r|$ versus ω^2. For what values of ω will resonance occur?