Pure Math Qualifying Exam: Sept. 11, 2004

Part I

1. Let $0 < b < a$. Use contour integration to evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{(a + b\cos(\theta))^2}.$$

2. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a real matrix with $a, b, c, d > 0$. Show that A has an eigenvector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, with $x, y > 0$.

3. Prove that every group of order p^m can be generated by m elements. Here p is a prime.

4. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}.$$

   (a) For which x does the series converge absolutely?

   (b) On which intervals does it converge uniformly?

   (c) Is f continuous wherever the series converges?

   (d) Is f bounded?

5. Suppose f is analytic on $D := \{z \in \mathbb{C} \mid |z| < 1\}$, continuous on its closure, $\bar{D}$, and real-valued on the boundary of D. Show that f is constant on $\bar{D}$.

6. Let $M^{n,n}$ denote the vector space of $n \times n$ real matrices. Consider the linear transformation $L : M^{n,n} \to M^{n,n}$ defined by $L(A) = A + A^T$ (here A^T denotes the transpose of the matrix A).

   (a) Let $n = 2$. Find bases for the kernel, $Ker(L)$, and the range, $Ran(L)$, of L.

   (b) For all $n \geq 2$, find the dimensions of $Ker(L)$ and $Ran(L)$.

Part II

1. Consider the vector field $\mathbf{F}(x, y, z) = (yz + x^4)\hat{\mathbf{i}} + (x(1+z) + e^y)\hat{\mathbf{j}} + (xy + \sin(z))\hat{\mathbf{k}}$. Let C be a circle of radius R lying in the plane $2x + y + 3z = 6$. What are the possible values of the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$?

2. Let $C = C^0([0,1])$ be the ring of continuous functions $f : [0,1] \longrightarrow \mathbf{R}$. For $a \in [0,1]$, define $I_a = \{f \in C \,|\, f(a) = 0\}$.

   (a) Show that I_a is a maximal ideal of C.

   (b) Show that every maximal ideal of C is of the form I_a for some $a \in [0,1]$.

   (c) Show that part (b) fails if the closed interval $[0,1]$ is replaced by the open interval $(0,1)$.

3. Show that $2e^{-z} - z + 3$ has exactly one root in the right half-plane $\{z \in \mathbb{C} \mid Re(z) > 0\}$.

4. Let a, b, c, d be real numbers, not all zero. Find the eigenvalues of the following 4×4 matrix and describe the eigenspace decomposition of $\mathbb{R}^4$:

$$\begin{pmatrix} aa & ab & ac & ad \\ ba & bb & bc & bd \\ ca & cb & cc & cd \\ da & db & dc & dd \end{pmatrix}$$

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5. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be continuous, and define $F(t) := \int_0^t f(s,t)ds$. Prove (carefully) that F is continuous on $[0,1]$.

6. Let F be a finite field, $f(x) \in F[x]$ be a polynomial with coefficients in F, and $F \subset E$ be a field extension (not necessarily finite). Show that if E contains one root of $f(x)$ then it contains every root of $f(x)$.