Pure Mathematics Qualifying Exam, September 6, 2003

Part I

1. Prove that the product of two uniformly continuous real-valued functions on $(0,1)$ is also uniformly continuous on $(0,1)$.

2. For what values of r and n is there an $n \times n$-matrix of rank r, with real entries, such that $A^2 = 0$? Here 0 denotes the $n \times n$ zero matrix.

3. Determine all entire functions $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ that satisfy $|f(z)| \leq e^{\mathrm{Re}(z)}$ for all complex z. (An entire function is one that is analytic for all complex z.)

4. Let G be a group, H be a subgroup of finite index n and $g \in G$.

   (a) Show that $g^k \in H$ for some $0 < k \leq n$.

   (b) Show by example that g^n may not lie in H.

5. Let $\phi : [0,1] \times \mathbb{R} \to \mathbb{R}$ be bounded and continuous. For each $n \in \mathbb{N}$ let $F_n : [0,1] \to \mathbb{R}$ satisfy

$$F_n(0) = \frac{1}{n}, \;\; F_n'(t) = \phi(t, F_n(t)) \text{ for } t \in [0,1].$$

   Here $F_n'(t)$ denotes the right derivative if $t = 0$ and the left derivative if $t = 1$.

   (a) Prove that there is a subsequence such that $\{F_{n_k}\}$ converges uniformly to a limit function F.

   (b) Prove that F solves

$$F(0) = 0, \;\; F'(t) = \phi(t, F(t)) \text{ for } t \in [0,1].$$

6. Show that there is no real $n \times n$ matrix A such that

$$A^2 = \begin{pmatrix} -a_1 & 0 & \dots & 0 \\ 0 & -a_2 & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & -a_n \end{pmatrix},$$

   where $a_1, \dots, a_n$ are distinct positive real numbers.

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Part II

7. Use contour integration to evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\sin \pi x}{x^2 + x + 1}\, dx$.

8. Let $\mathbb{Z}$ be the ring of integers, p a prime, and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ the field with p elements. Let x be an indeterminant, and set $R_1 = \mathbb{F}_p[x]/(x^2 - 2)$, $R_2 = \mathbb{F}_p[x]/(x^2 - 3)$. Determine whether the rings R_1 and R_2 are isomorphic in each of the following cases:

   (a) $p = 2$,

   (b) $p = 5$,

   (c) $p = 11$.

9. Let $\mathbf{C}$ be a simple closed C^1-curve in $\mathbb{R}^2$ with the positive orientation enclosing a region D. Assume D has area 2 and centroid $(3, 4)$. Let $\mathbf{F}(x, y) = (y^2, x^2 + 3x)$. Find the line integral $\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{s}$.

10. Let A be a nilpotent $n \times n$-matrix, i.e., $A^m = 0$ for some $m \geq 1$, where 0 is the $n \times n$-zero matrix. Prove or disprove the following assertions:

    (a) $A^n = 0$,

    (b) $\det(A + I) = 1$. Here I denotes the $n \times n$ identity matrix.

    (c) $\det(D + A) = \det(D)$ for every diagonal $n \times n$-matrix D?

11. (a) Show that all the zeros of the polynomial $f(z) = z^8 - 3z + 1$ lie in the disk $|z| < 5/4$. (b) How many zeros does f have in the unit circle?

12. A complex number is called *algebraic* if it is a root of a non-zero polynomial with integer coefficients. Show that $a = \sin(r^o)$ is an algebraic number for every rational number r. Here r^o denotes the angle of r degrees or, equivalently, of $\frac{\pi r}{180}$ radians.