Applied Mathematics Qualifying Exam, September 6, 2003

Part I

1. For what values of r and n is there an $n \times n$-matrix of rank r, with real entries, such that $A^2 = 0$? Here 0 denotes the $n \times n$ zero matrix.

2. Show that there is no real $n \times n$ matrix A such that
$$A^2 = \begin{pmatrix} -a_1 & 0 & \dots & 0 \\ 0 & -a_2 & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & -a_n \end{pmatrix},$$
where $a_1, \dots, a_n$ are distinct positive real numbers.

3. Define $\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}$ to be the trace of the complex $n \times n$ matrix $A = (a_{ij})$. Prove that

   (a) $\mathrm{Tr}\left(BAB^{-1}\right) = \mathrm{Tr}(A)$ for any invertible matrix B.

   (b) $\mathrm{Tr}(A) = \sum_{i=1}^{n} \lambda_i$, where λ_i for $i = 1, .., n$ are the eigenvalues of A repeated according to multiplicity.

4. Consider the Fourier series of the real-valued function f on the interval $[-\pi, \pi]$ of the form:
$$a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

   (a) Suppose that $f(x)$ is differentiable on $[-\pi, \pi]$, $f(-\pi) = f(\pi)$, and $f^{'}(x)$, $f^{''}(x)$ are piecewise continuous, with jump discontinuities. Then, stating carefully any theorems you may use, show that
$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f^{'}(x)|^2 \, dx = \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2) \, .$$

   (b) Next, suppose that $f(x)$ has two continuous derivatives on $[-\pi, \pi]$. Show that its Fourier Cosine coefficients obey the bound $|a_n| < C/n^2$ for some appropriate constant C.

5. The surface $\mathcal{S}$ is defined by $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, where $0 < a < b < c < 1$. Let $\mathcal{Q} = (0, 1, 1)$. Find the point $\mathcal{P}$ on $\mathcal{S}$ that is closest to $\mathcal{Q}$.

6. Determine all entire functions $f \colon \mathbb{C} \longrightarrow \mathbb{C}$ that satisfy $|f(z)| \leq e^{\mathrm{Re}(z)}$ for all complex z. (An entire function is one that is analytic for all complex z.)

Applied Mathematics Qualifying Exam, September 6, 2003

Part II

7. Use contour integration to evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\sin \pi x}{x^2+x+1}\,dx$.

8. (a) Show that all the zeros of the polynomial $f(z) = z^8 - 3z + 1$ lie in the disk $|z| < 5/4$.
(b) How many zeros does f have in the unit circle?

9. Let $\mathbf{C}$ be a simple closed C^1-curve in $\mathbb{R}^2$ with the positive orientation enclosing a region D. Assume D has area 2 and centroid $(3,4)$. Let $\mathbf{F}(x,y) = (y^2, x^2+3x)$. Find the line integral $\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{s}$.

10. Consider the following heat equation for $u(x,y,t)$ in two spatial dimensions:

$$u_t = D_1 u_{xx} + D_2 u_{yy}\,, \qquad -\infty < x < \infty\,, \quad -\infty < y < \infty\,, \quad t > 0\,,$$
$$u(x,y,0) = \delta(x)\delta(y)\,.$$

Here $\delta(x)$ denotes the Dirac delta function. In addition, $D_1 > 0$ and $D_2 > 0$ are constants. Assuming that $u(x,y,t) \to 0$ as $x^2+y^2 \to \infty$, calculate the solution using Fourier Transforms. For a fixed value of t, what are the curves of constant u in the (x,y) plane?

11. Consider the following radially symmetric heat equation for $u = u(r,t)$ in an insulated sphere of radius R with $R > 0$:

$$u_t = D\left(u_{rr} + \frac{2}{r}u_r\right)\,, \qquad 0 < r < R\,, \quad t > 0\,,$$
$$u(r,0) = u_0\left(\frac{r}{R}\right)^2\,; \qquad u_r(R,t) = 0\,; \quad \text{with} \quad u \quad \text{bounded} \quad \text{as} \quad r \to 0\,.$$

Here $D > 0$ and $u_0 > 0$ are constants.

(a) Non-dimensionalize the problem.

(b) Calculate the steady-state $\lim_{t\to\infty} u(r,t)$.

(c) Derive an approximation for u valid for long time that shows the approach of u to the steady-state solution. (Hint: The substitution $v(r) = f(r)/r$ in $v_{rr} + (2/r)\,v_r + \lambda v = 0$ yields a simple equation for v.)

12. A model for the outbreak of an insect infestation in the presence of predators is

$$\frac{dN}{dt} = RN\left(1 - \frac{N}{K}\right) - P(N)\,.$$

Here $R > 0$ and $K > 0$ are constants, N is the population of insects at time t, and the predation term $P(N)$ is

$$P(N) = \frac{BN^2}{A^2 + N^2}\,,$$

where $A > 0$ and $B > 0$ are constants.

(a) Non-dimensionalize the model to the form

$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{\kappa}\right) - \frac{x^2}{1 + x^2}\,. \tag{4}$$

(b) Graphically determine the equilibrium solutions for (4).

(c) Show that for a fixed r not too small, there is a range of values of k where there are multiple stable steady-state solutions for (4).