Analysis Qualifying Exam (Draft)
(January XX, 2014)

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1. (10 points) Prove that the series
$$\sum_{n=1}^{\infty} \frac{x^2 \cos(n^2 x)}{n^2}$$
converges pointwise to a continuous function on $\mathbb{R}$.

Solution: On any bounded interval $[-R, R]$, we have
$$\left| \frac{x^2 \cos(n^2 x)}{n^2} \right| \leq \frac{R^2}{n^2}$$
and the series $\sum_{n=1}^{\infty} \frac{R^2}{n^2}$ is convergent. Hence the original series is uniformly convergent on any interval $[-R, R]$, and the summands are continuous, so that the series is pointwise convergent to a continuous function on $[-R, R]$. Since R was arbitrary, the limit is continuous on $\mathbb{R}$.

2. (10 points) Use Green's theorem to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = e^{x^2}\mathbf{i} + e^{2x+y}\mathbf{j}$ and C is the boundary of the rectangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(0,1)$, $(2,1)$ and $(2,0)$, oriented clockwise.

Solution:
$\int_C \mathbf{F} \cdot d\mathbf{r} = -\iint_D (2e^{2x+y} - 0) dA$, where D is the rectangle as above. (The minus sign is because C is negatively oriented.) So
$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_0^2 \int_0^1 2e^{2x+y} dy dx = -\int_0^2 2e^{2x} dx \int_0^1 e^y dy = -\left(e^{2x}|_0^2\right)\left(e^y|_0^1\right) = -(e^4 - 1)(e - 1).$$

3. (10 points) Assume that a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $[a, b]$, $f(a) = 0$, and that there exists a constant $C \geq 0$ such that $|f'(x)| \leq C|f(x)|$ for $x \in [a, b]$. Prove that $f(x) \equiv 0$ on $[a, b]$.

Solution: Suppose that $f \not\equiv 0$, then there is a $x \in (a, b]$ such that $f(x) \neq 0$. Without loss of generality, we may assume that $f(x) > 0$. Since f is continuous, we can choose an interval $(a_1, b_1) \subset [a, b]$ such that $f(x) > 0$ on (a_1, b_1). Taking a_1 as small as possible, we may further assume that $a_1 = 0$ (note that $a_1 \geq a$ since $f(a) = 0$.) Let $a_1 < x < y < b_1$, then by the mean value theorem
$$|\ln f(y) - \ln f(x)| = |(\ln f)'(\theta)| = \frac{|f'(\theta)|}{|f(\theta)|}$$
for some $\theta \in (x, y)$. By the assumption on f, we get that $|\ln f(y) - \ln f(x)| \leq C$ for all $x, y \in (a_1, b_1)$. But if $x \searrow a_1$, then $f(x) \searrow 0$, so that $\ln f(x) \to \infty$ and $|\ln f(y) - \ln f(x)| \to \infty$ for any fixed y, a contradiction.

4. (20 points). Evaluate the integral $\displaystyle\int_0^{\infty} \frac{x^{\alpha}}{1 + x + x^2} dx$ where $0 < \alpha < 1$.

Solution: We use a branch cut for z^α; we take this along the positive real axis and define

$$z^\alpha = r^\alpha e^{i\alpha\theta}$$

where $z = re^{i\theta}$ and $0 \le \theta < 2\pi$.

Consider

$$\int_C \frac{z^\alpha}{1+z+z^2} dz$$

where the keyhole contour C consists of a large circle C_R of radius R, a small circle C_ϵ of radius ϵ (to avoid the singularity of z^α at $z = 0$) and two lines just above and below the branch cut.

The contribution from C_R is $O(R^{\alpha-2}) \times 2\pi R = O(R^{\alpha-1}) \to 0$ as $R \to +\infty$.

The contribution from C_ϵ is (substituting $z = \epsilon e^{i\theta}$ on C_ϵ)

$$\int_{2\pi}^{0} \frac{\epsilon^\alpha e^{i\alpha\theta}}{1+\epsilon e^{i\theta}+\epsilon^2 e^{2i\theta}} i\epsilon e^{i\theta} d\theta = O(\epsilon^{\alpha+1}) \to 0$$

The contribution from just above the branch cut is

$$\int_\epsilon^R \frac{x^\alpha}{1+x+x^2} dx \to I$$

as $\epsilon \to 0$ and $R \to +\infty$. The contribution from just below the branch cut is

$$\int_R^\epsilon \frac{x^\alpha e^{2\alpha\pi i}}{1+x+x^2} dx \to -e^{2\alpha i} I$$

as $\epsilon \to 0$ and $R \to +\infty$.

Hence

$$\int_C \frac{z^\alpha}{1+z+z^2} dz \to (1-e^{2\pi\alpha i}) I$$

as $\epsilon \to 0$ and $R \to \infty$.

But the integrand is equal to

$$\frac{z^\alpha}{(z-e^{\frac{2}{3}\pi i})(z-e^{\frac{4\pi}{3}i})}$$

so the poles inside C are at $e^{\frac{2\pi}{3}i}$ with residue $\frac{e^{\frac{2\alpha\pi}{3}i}}{i}$ and at $e^{\frac{4\pi}{3}i}$ with residue $\frac{e^{\frac{4\alpha\pi}{3}i}}{-i}$.

We conclude that

$$(1-e^{2\pi\alpha i}) I = 2\pi i \left(\frac{e^{\frac{2\alpha\pi}{3}i}}{i} + \frac{e^{\frac{4\alpha\pi}{3}i}}{-i}\right)$$

$$I = 2\pi \frac{\sin\frac{\alpha\pi}{3}}{\sin(\alpha\pi)}$$

5. (20points) (a) (10points) Use Rouche's theorem to prove the Fundamental Theorem of Algebra: every non-zero, single-variable, degree n polynomial with complex coefficients has, counted with multiplicity, exactly n roots.

(b) (10points) How many zeroes does the function $f(z) = z^{20} + 4z^2e^{z+1} - 3z^8$ have in the unit disk $\{|z| < 1\}$?

Solution:

Solution to (a): Let $P_n(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_0$ and $C = \{|z| = R\}$ where R is large. Now choose

$$F(z) = z^n, \quad G(z) = a_{n-1}z^{n-1} + ... + a_0$$

On C, $|G(z)| \leq |a_{n-1}||R|^{n-1} + |a_{n-2}|R^{n-2} + ... + |a_0| < |a_n|R^n$ if R is sufficiently large.

Since F has n zeroes (counting multiplicity), by Rouche's theorem, $F + G$ has exactly n zeroes in $\{|z| < R\}$.

One should also prove that for $|z| = R$ large there are no zeroes.

Solution to (b): We take

$$F(z) = 4z^2e^{z+1}, G(z) = z^{20} - 3z^8$$

and estimate on the circle $|z| = 1$

$$|G(z)| \leq |z|^{20} + 3 \leq 4, |F(z)| = 4e^{Re(z)+1} \geq 4$$

A more closer look shows that

$$|G(z)| < |F(z)|$$

Since the function $F(z)$ has two zeroes in $\{|z| < 1\}$, by Rouche's theorem, $f(z) = z^{20} + 4z^2e^{z+1} - 3z^8$ also has two zeroes in the unit disk $\{|z| < 1\}$.

6. (20pints) (a) (10points) Classify all analytic functions having the property that

$$f(z + m + ni) = f(z) \quad (z \in \mathbb{C}, m, n \in \mathbb{Z})$$

where $\mathbb{Z}$ denotes the set of integers.

(b) (10points) Let $\Omega = \{z \in \mathbb{C} \mid \frac{3}{4}\pi < |z| < \frac{7}{4}\pi\}$. Show that there does not exist a sequence $\{P_n(z)\}$ of polynomials in z such that $P_n(z) \to \tan(z)$ uniformly in any compact set in Ω.

Solution:

Solution to (a): We claim that f must be constant. In fact, let $S = [0, 1] \times [0, 1]$. The perioidicity condition on f gives that $f(\mathbb{C}) = f(S)$. Since S is compact and f is continuous (it is holomorphic), it follows that f is bounded on S, and therefore, f is bounded on $\mathbb{C}$. By Liouville's Theorem, we deduce that f is constant.

Solution to (b): We prove it by contradiction. Suppose that there does exist a sequence $\{P_n(z)\}$ of polynomials in z such that $P_n(z) \to \tan(z)$ uniformly in any compact set in Ω. In particular, we take

$$C = \{|z| = \pi\}$$

By Cauchy residue theorem,

$$\int_C P_n(z)dz = 0$$

By uniform convergence we then have

$$\int_C \tan(z)dz = 0$$

But $\tan(z)$ has two poles $z = \frac{\pi}{2}, -\frac{\pi}{2}$ inside $\{|z| < \pi\}$ with residue -1 and hence

$$\int_C \tan(z)dz = -4\pi i$$

This reaches a contradiction.