Analysis Qualifying Exam

University of British Columbia

January 5, 2013

1. (a) Find the volume of the solid given by $x^2 + z^2 \le 1$, $y^2 + z^2 \le 1$.

   (b) Let $\{x_i\}_{i=1}^{\infty}$ be an infinite sequence of real numbers such that every subsequence contains a subsequence converging to 0. Must the original sequence converge?

2. For each of the following statements, either prove it or give a counterexample:

   (a) If the functions $h_n(x)$ are continuous real-valued functions on $[0,1]$ such that $\lim_{n\to\infty} h_n(x) = h(x)$ for all $x \in [0,1]$, then $h(x)$ is continuous.

   (b) If $f(x)$ and $f_n(x)$ are continuous real-valued functions on $[0,1]$, and $\lim_{n\to\infty} f_n(x) = f(x)$ for almost all $x \in [0,1]$, then $\lim_{n\to\infty} \int_0^1 f_n(x) = \int_0^1 f(x)$.

   (c) If $g(m,n)$ is real for all integers m,n and $\sum_{m=0}^{\infty}(\sum_{n=0}^{\infty} g(m,n))$ and $\sum_{n=0}^{\infty}(\sum_{m=0}^{\infty} g(m,n))$ are both defined, then they are equal.

3. Prove that the sequence of functions $f_n(x) = \sin(nx)$ has no pointwise convergent subsequence.

4. (a) Compute the integral
$$\int_\Gamma \frac{e^z}{z^2 + a^2} dz$$
   where Γ is the circle $|z| = 2a$, $a > 0$, oriented counter-clockwise.

   (b) Compute the integral
$$\int_\Gamma \frac{ze^z}{(z-b)^3} dz$$
   where Γ is a simple closed smooth loop in $\mathbb{C}$ with counter-clockwise orientation and $b \notin \Gamma$.

5. Let $f(z) = \sqrt{|xy|}$, where $z = x + iy \in \mathbb{C}$. Does f satisfy the Cauchy-Riemann equations at $z = 0$? Does $f'(0)$ exist?

6. Suppose the radius of convergence of the power series $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ is 1 and f has only finitely many singularities $z_1, \cdots, z_m$ on the unit circle $C = \{z \in \mathbb{C} : |z| = 1\}$ which are all simple poles. Show that the sequence $\{a_n\}$ is bounded.