Mathematics Qualifying Exam
University of British Columbia
January 14, 2012

Part I: Real and Complex Analysis (Pure and Applied Exam)

1. (a) Find all polynomials that are uniformly continuous on $\mathbb{R}$.

   (b) Let A be a nonempty subset of $\mathbb{R}$ and let f be a real-valued function defined on A. Further let $\{f_n\}$ be a sequence of bounded functions on A which converge uniformly to f. Prove that

$$\frac{f_1(x) + \cdots + f_n(x)}{n} \to f(x)$$

   uniformly on A as $n \to \infty$.

2. (a) Prove the Logarithmic Test

   Theorem 1. *Suppose that $a_k \neq 0$ for large k and that*

$$p = \lim_{k\to\infty} \frac{\log(1/|a_k|)}{\log k} \textit{ exists.}$$

   - *If $p > 1$ then $\sum_{k=1}^{\infty} a_k$ converges absolutely, and*
   - *If $p < 1$ then $\sum_{k=1}^{\infty} |a_k|$ diverges.*

   (b) Let $\{a_k\}$ be a sequence of non-zero real numbers and suppose that

$$p = \lim_{k\to\infty} k\left(1 - \left|\frac{a_{k+1}}{a_k}\right|\right) \text{ exists}$$

   Prove that $\sum_{k=1}^{\infty} a_k$ converges absolutely when $p > 1$.

3. Evaluate the integral

$$\iint_S \mathbf{F} \cdot \mathbf{n} \mathrm{d}\sigma,$$

   where S is the region of the plane $y = z$ lying inside the unit ball centred at the origin, and $\mathbf{F} = (xy, xz, -yz)$, and $\mathbf{n}$ is the upward-pointing normal.

   Note that it might be helpful to remember that

$$\int 2\sin^2 t \mathrm{d}t = t - \sin t \cos t.$$

4. In the following, justify your answer.

   (a) (6 points) Prove or disprove:
   There exists a holomorphic function f on $\mathbb{C}$ (thus an entire function) such that $f(D) = Q$ where D is the unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and Q is the square $Q = \{z \in \mathbb{C} \mid -1 < \operatorname{Re} z, \operatorname{Im} z < 1\}$.

   (b) (7 points) Find all holomorphic functions $f(z)$ on $\mathbb{C} \setminus \{0\}$ such that

$$f(1) = 1, \qquad |f(z)| \leq \frac{1}{|z|^3}$$

   (c) (7 points) Find a holomorphic function $f(z)$ on $D = \{z \in \mathbb{C} \mid |z| < 1\}$, which maps D onto the infinite sector

$$S = \{z = re^{i\theta} \in \mathbb{C} \mid 0 < \theta < \pi/4\}.$$

5. (a) (6 points) Prove or disprove:
   There exists a **nonconstant** holomorphic function $f(z)$ from $D = \{z \in \mathbb{C} \mid |z| < 1\}$ into $\mathbb{C}$ such that the area of its image, area $f(D) = 0$.

   (b) (7 points) Show that there is **no** holomorphic function $f(z)$ on $D = \{z \in \mathbb{C} \mid |z| < 1\}$ such that $|f(z)| = |z|^{1/2}$ for all $z \in D$.

   (c) (7 points) Find all harmonic functions $u(x, y)$ on $\mathbb{R}^2$ such that $e^{u(x,y)} \leq 10 + (x^2 + y^2)$ and $u(1, 1) = 0$.

6. (20 points) Evaluate the following integral, using contour integration, carefully justifying each step:

$$\int_0^\infty \frac{\log x}{(1 + x^2)^2} dx$$

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Part II: Linear Algebra and Algebra (pure exam)

1. Determine the eigenvalues and a basis of the corresponding eigenspaces for the linear map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix $\mathbf{A}$ with respect to the standard basis, where:

$$\mathbf{A} = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

   Note: all eigenvalues are rational numbers.

2. Let $\mathcal{N}_n \subset M_n(\mathbb{R})$ be the set of *nilpotent* matrices, that is the set of $n \times n$ matrices A such that $A^k = 0$ for some k. Show that $\mathcal{N}_n$ is a closed subset of $M_n(\mathbb{R})$ (identify the latter with $\mathbb{R}^{n^2}$).

3. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.
   (a) Show that there is a unique integer $0 \leq k \leq \min\{n, m\}$ for which there are bases $\{\underline{u}_i\}_{i=1}^n \subset \mathbb{R}^n$ $\{\underline{v}_i\}_{i=1}^m \subset \mathbb{R}^m$ such that the matrix of T with respect to these bases is $D^{(k)}$, where

$$D^{(k)} = \begin{cases} 1 & 1 \leq i = j \leq k \\ 0 & \text{otherewise} \end{cases},$$

   that is $D^{(k)}$ has zeroes everywhere except that the first k entries on the main diagonal are 1.

   (b) Show that the row rank and column rank of any matrix $A \in M_{m,n}(\mathbb{R})$ are equal.

4. (a) Suppose that the order of a finite group G is divisible by 3 but not 9. Show that there are either one or two conjugacy classes of elements of order 3 in G.

   (b) Give examples of finite groups A, B, C of order divisible by 3 so that the orders of A, B are not divisible by 9 and they have one and two conjugacy classes of elements of order 3, respectively, and so that the order of C is divisible by 9 and it has more than two such conjugacy classes.

5. (a) Let R be an integral domain, and let $f \in R[x]$ be a polynomial. Let $\{a_i\}_{i=1}^r \subset R$ be distinct, and suppose that $f(a_i) = 0$ for all i. Show that $\prod_{i=1}^r (x - a_i)$ divides f in $R[x]$.

   (b) Let $\{a_i\}_{i=1}^n, \{b_j\}_{j=1}^n$ be algebraically indepedent, and let $F = \mathbb{Q}(\underline{a}, \underline{b})$ be the field of rational functions in $2n$ variables over $\mathbb{Q}$. Let $A \in M_n(F)$ be the matrix where $A_{ij} = \frac{1}{a_i - b_j}$. Show that

$$\det A = c_n \frac{\prod_{1 \leq i < j \leq n} ((a_i - a_j)(b_i - b_j))}{\prod_{i=1}^n \prod_{j=1}^n (a_i - b_j)}$$

for some universal $c_n \in \mathbb{Q}$.

For $n = 2$ this identity is:

$$\det \begin{pmatrix} \frac{1}{a_1 - b_1} & \frac{1}{a_1 - b_2} \\ \frac{1}{a_2 - b_1} & \frac{1}{a_2 - b_2} \end{pmatrix} = -\frac{(a_1 - a_2)(b_1 - b_2)}{(a_1 - b_1)(a_1 - b_2)(a_2 - b_1)(a_2 - b_2)}.$$

6. Let $f(x) = x^6 + 5x^3 + 1 \in \mathbb{Q}[x]$.

   (a) Construct a splitting field Σ for f by adjoining at most two elements to $\mathbb{Q}$. You may wish to use the primitive cube root of unity $\omega = \frac{-1+\sqrt{-3}}{2}$.

   (b) Given that f has no root in $\mathbb{Q}(\sqrt{-3}, \sqrt{21})$ find $[\Sigma : \mathbb{Q}]$ and show that f is irreducible in $\mathbb{Q}[x]$.

   (c) Let $\beta \in \Sigma$ be a root of F. Show that there exist unique $\rho, \sigma \in \mathrm{Gal}(\Sigma : \mathbb{Q})$ so that: $\rho(\beta) = \frac{1}{\beta}$, $\rho(\omega) = \omega$, $\sigma(\beta) = \beta$, $\sigma(\omega) = \omega^2$. Also, show that ρ and σ commute.