Mathematics Qualifying Exam
University of British Columbia
January 8, 2011

Part I: Real and Complex Analysis (Pure and Applied Exam)

1. Assume that $f : [0, 1] \to \mathbb{R}$ is a smooth function. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x)e^{inx^3}\, dx = 0.$$

2. a) Assume $f(x)$ is a strictly increasing continuous function with $f(0) = 0$ and with inverse f^{-1}. Show that

$$\int_0^a f(x)\, dx + \int_0^b f^{-1}(x)\, dx \geq ab,$$

for any two positive real numbers a and b. For what b does the equality hold?

b) Use this to prove Young's inequality, which states that if p and q are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ then

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab.$$

3. a) Find a counter example to the following statement:
If $f_n(x)$ for $n > 0$ is a sequence of continuous real-valued functions on the unit interval $[0, 1]$ such that $\lim_{n\to\infty} f_n(x) = 0$ for all x. Then

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0.$$

b) Find a minimal extra condition that can not be simplified, which makes this statement true.

Please turn over

4. Use a contour integral to evaluate

$$\int_0^\infty \frac{d\,x}{1+x^{2n}}, \qquad n \geq 1.$$

5. a) Show by contour integration that

$$\int_0^{2\pi} \frac{d\,\theta}{x+\cos\theta} = \frac{2\pi}{\sqrt{x^2-1}}, \qquad \text{if } x > 1.$$

b) Determine for which complex values of w, the function $f(w)$ defined as

$$f(w) = \int_0^{2\pi} \frac{d\,\theta}{w+\cos\theta}$$

is analytic. Evaluate the integral for those w. Simplify your answer as much as possible. Justify your reasoning with all details.

6. Consider the meromorphic function

$$f(z) = \frac{1-z^2}{2i(z^2-(a+\frac{1}{a})z+1)}, \qquad |a| < 1.$$

Find the Laurent series expansion for $f(z)$ valid in a neighborhood of the unit circle $|z| = 1$.

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Part II: Linear Algebra and Algebra (Pure Exam)

1. a) Over the vector space $\mathcal{P}$ of all polynomials we consider the inner product

$$\langle P, Q \rangle = \int_0^1 P(x)Q(x)\,dx.$$

Find a polynomial of degree 2 that is orthogonal to $P_0(x) = 1$ and $P_1(x) = x$.

b) Let now $\mathcal{P}_n$ be the vector space of the polynomials of degree less or equal than n. Consider the linear mapping $\mathcal{F} : \mathcal{P}_n \to \mathcal{P}_n$ defined by

$$\mathcal{F}(P)(x) = (x-1)P'(x),$$

for $P \in \mathcal{P}_n$. Find the matrix F that describes $\mathcal{F}$ with respect to the basis $\{1, x, x^2, \ldots, x^n\}$.

2. Let A be an $n \times n$ matrix with real coefficients. Show the following:

a) If the sum of the elements in each of the columns of A is 1, then $\lambda = 1$ is an eigenvalue of A.

b) If A is invertible and v is an eigenvector of A, then v is also an eigenvector of both A^2 and A^{-2}. What are the corresponding eigenvalues?

c) If $AB = BA$ for all invertible matrices B, then $A = cI$ for some scalar c.

3. a) Let A be an $n \times m$ matrix with real coefficients. Let v_i denote the i-th row of A, and let B be the matrix obtained from A by the elementary row operation which replaces v_j with $v_j - av_i$, for $a \in \mathbb{R}$ and $i \neq j$. Thus the rows w_i of B are given by $w_i = v_i$ if $i \neq j$, and $w_j = v_j - av_i$. Then show that there exists an invertible $n \times n$ matrix E such that $B = EA$.

b) Use part a) to show that the rank of the row space of A is equal to the rank of the column space of A.

Please turn over

4. a) Let K be a field and let $f(X), g(X)$ be monic irreducible polynomials with coefficients in K. Suppose there exists an extension L/K and an element $\alpha \in L$ such that $f(\alpha) = g(\alpha) = 0$. Then show that $f = g$.

   b) Let $f(X) = X^n + a_{n-1}X^{n-1} + \ldots a_0$ be a monic polynomial with rational integer coefficients. Suppose there exists a rational number α with $f(\alpha) = 0$. Then show that α is a rational integer.

   c) Show that the polynomial $X^4 + 1$ is irreducible in $\mathbf{Q}[X]$.

   d) Find the Galois group over $\mathbf{Q}$ of the polynomial $X^3 - 2$.

5. a) Let R be a commutative ring (with identity element) and let I and J be ideals of R. Show that the set $I + J = \{i + j | i \in I, j \in J\}$ is an ideal of R.

   b) With the notations of part a), suppose that $I + J = R$. Then show that R/IJ is isomorphic to $R/I \oplus R/J$.

6. a) Let G be a finite group. If $x \in G$, let G_x denote the set of elements in G that are conjugate to x, namely, $z \in G_x \iff \exists y \in G$ with $yxy^{-1} = z$. Show that the cardinality of the set G_x divides the order of G.

   b) If the group G has order p^r where p is a prime, then show that there exists some $x \neq 1 \in G$ such that $G_x = \{x\}$.