

Mathematics Qualifying Exam
University of British Columbia
January 8, 2011

Part I: Real and Complex Analysis (Pure and Applied Exam)

1. Assume that $f : [0, 1] \rightarrow \mathbb{R}$ is a smooth function. Prove that

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) e^{inx^3} dx = 0.$$

2. a) Assume $f(x)$ is a strictly increasing continuous function with $f(0) = 0$ and with inverse f^{-1} . Show that

$$\int_0^a f(x) dx + \int_0^b f^{-1}(x) dx \geq ab,$$

for any two positive real numbers a and b . For what b does the equality hold?

- b) Use this to prove Young's inequality, which states that if p and q are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ then

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab.$$

3. a) Find a counter example to the following statement:

If $f_n(x)$ for $n > 0$ is a sequence of continuous real-valued functions on the unit interval $[0, 1]$ such that $\lim_{n \rightarrow \infty} f_n(x) = 0$ for all x . Then

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = 0.$$

- b) Find a minimal extra condition that can not be simplified, which makes this statement true.

Please turn over

4. Use a contour integral to evaluate

$$\int_0^{\infty} \frac{dx}{1+x^{2n}}, \quad n \geq 1.$$

5. a) Show by contour integration that

$$\int_0^{2\pi} \frac{d\theta}{x + \cos \theta} = \frac{2\pi}{\sqrt{x^2 - 1}}, \quad \text{if } x > 1.$$

b) Determine for which complex values of w , the function $f(w)$ defined as

$$f(w) = \int_0^{2\pi} \frac{d\theta}{w + \cos \theta}$$

is analytic. Evaluate the integral for those w . Simplify your answer as much as possible. Justify your reasoning with all details.

6. Consider the meromorphic function

$$f(z) = \frac{1 - z^2}{2i(z^2 - (a + \frac{1}{a})z + 1)}, \quad |a| < 1.$$

Find the Laurent series expansion for $f(z)$ valid in a neighborhood of the unit circle $|z| = 1$.

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Part II: Linear Algebra and Differential Equations (Applied Exam)

1. a) Over the vector space \mathcal{P} of all polynomials we consider the inner product

$$\langle P, Q \rangle = \int_0^1 P(x)Q(x) dx.$$

Find a polynomial of degree 2 that is orthogonal to $P_0(x) = 1$ and $P_1(x) = x$.

- b) Let now \mathcal{P}_n be the vector space of the polynomials of degree less or equal than n . Consider the linear mapping $\mathcal{F} : \mathcal{P}_n \rightarrow \mathcal{P}_n$ defined by

$$\mathcal{F}(P)(x) = (x - 1)P'(x),$$

for $P \in \mathcal{P}_n$. Find the matrix F that describes \mathcal{F} with respect to the basis $\{1, x, x^2, \dots, x^n\}$.

2. Let A be an $n \times n$ matrix with real coefficients. Show the following:

- a) If the sum of the elements in each of the columns of A is 1, then $\lambda = 1$ is an eigenvalue of A .
- b) If A is invertible and v is an eigenvector of A , then v is also an eigenvector of both A^2 and A^{-2} . What are the corresponding eigenvalues?
- c) If $AB = BA$ for all invertible matrices B , then $A = cI$ for some scalar c .

3. a) Let A be an $n \times m$ matrix with real coefficients. Let v_i denote the i -th row of A , and let B be the matrix obtained from A by the elementary row operation which replaces v_j with $v_j - av_i$, for $a \in \mathbb{R}$ and $i \neq j$. Thus the rows w_i of B are given by $w_i = v_i$ if $i \neq j$, and $w_j = v_j - av_i$. Then show that there exists an invertible $n \times n$ matrix E such that $B = EA$.
- b) Use part a) to show that the rank of the row space of A is equal to the rank of the column space of A .

Please turn over

4. Suppose the motion of a certain mass-spring system satisfies the differential equation

$$u'' + \gamma u' + 4u = 0, \quad u(0) = 1, \quad u'(0) = a, \quad (1)$$

with units m, kg, and s. Here $u(t)$ is the displacement from the equilibrium position, $\gamma > 0$ is the damping coefficient, and a is a real parameter.

- a) Determine the range of γ so that the solution $u(t)$ is not oscillatory.
- b) Suppose $\gamma = 5$. Determine the range of real a so that the solution of (1) returns to the equilibrium point $u = 0$ at some finite $t > 0$.

5. Consider the nonlinear system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + x^3. \quad (2)$$

- a) Find the critical points for the system (2). Classify each critical point using the linearized system around it, specifying both the type of critical point and whether it is (asymptotically) stable or unstable. If a critical point cannot be unambiguously classified using the linearized system, please indicate this in your answer.
- b) Let $I(x, y) = x^2 + y^2 - \frac{1}{2}x^4$. Show that

$$I(x, y) \leq x^2 + y^2 \leq 2I(x, y)$$

if $x^2 + y^2$ is sufficiently small. Also show that $I(x(t), y(t))$ is constant in t for any solution $(x(t), y(t))$.

- c) Discuss the stability of the critical point $(0, 0)$ using b).

6. a) Solve the damped wave equation

$$\begin{aligned} u_{tt} + 2u_t &= u_{xx}, & 0 < x < \pi, \quad t > 0 \\ u(0, t) &= 0, \quad u(\pi, t) = 0 \\ u(x, 0) &= f(x), \quad u_t(x, 0) = 0, & 0 < x < \pi. \end{aligned}$$

b) Consider the following Laplace equation with derivative boundary conditions:

$$\begin{aligned} u_{xx} + u_{yy} &= 0, & 0 < x < a, \quad 0 < y < b, \\ u_x(0, y) &= 0, \quad u_x(a, y) = f(y), \\ u_y(x, 0) &= 0, \quad u_y(x, b) = 0. \end{aligned}$$

- i) In general, this problem does not have a solution. Place a condition on $f(y)$ so that the problem has solutions.
- ii) Then find the general form of the solution $u(x, y)$.