**UBC Math Qualifying Exam January 9th 2010: Morning Exam, 9am-12pm**
**PURE and APPLIED exam**

**Solve all six problems, and start each problem on a new page.**

1. Prove or disprove: The series
$$\sum_{n=1}^{\infty} \frac{\sin x}{1+n^2x^2}$$
converges uniformly on $[-\pi, \pi]$.

2. Let $Q = \{0 < x < 1,\ 0 < y < 1\}$. For what values of $a$, $b$ is the function
$$x^a y^b \int_0^\infty \frac{1}{(x+t)(y^2+t^2)}\, dt$$
bounded on $Q$?

3. (a) Does $p_N = \prod_{n=2}^{N}(1 + \frac{(-1)^n}{n})$ converge to a nonzero limit as $N \to \infty$? Explain your answer!
   (b) Prove that $\int_0^\infty \cos(t^2)dt$ converges.

4. Let $f(z) = \int_0^\infty e^{-zt^2} dt$.
   (a) Show that $f(z)$ is analytic in the domain $\mathrm{Re}(z) > 0$.
   (b) Assume that $f(1) = \frac{1}{2}\sqrt{\pi}$. Find the analytic continuation of $f(z)$ into the domain $\mathbb{C} \sim (-\infty, 0]$. This is the domain that is the whole complex plane except the negative real axis and zero. [Hint: compute $f(x)$ for $x > 0$.]
   (c) Let $F(z)$ denote the analytic continuation of $f(z)$ from part (b). Evaluate $F(i)$.
   (d) Evaluate $\int_0^\infty \cos(t^2)dt$.

5. Calculate the following integrals:
   (a)
$$\int_C (\bar{z})^2\, dz$$
   where $C$ is the circle $|z+1| = 4$, oriented counterclockwise.
   (b)
$$\int_C z\sin(z^{-1})dz$$
   where $C$ is the circle $|z| = 100$, oriented counterclockwise.
   (c)
$$\int_C \frac{\sin 3z}{(z-1)^4} dz$$
   where $C$ is the circle $|z| = 2$, oriented counterclockwise.

6.
$$J = \int_0^\infty \frac{(\ln x)^2}{x^2+9}\, dx.$$
Evaluate $J$, explaining all steps and calculations carefully.

**UBC Math Qualifying Exam January 9th 2010: Afternoon Exam, 1pm-4pm**
**PURE exam**

**Solve all eight problems, and start each problem on a new page.**

1. Let $\mathbf{x}$ be a unit vector in $\mathbb{R}^n$ and let $A = I - \beta \mathbf{x}\mathbf{x}^T$.
   (a) Show that $A$ is symmetric.
   (b) Find all values of $\beta$ for which $A$ is orthogonal.
   (c) Find all values of $\beta$ for which $A$ is invertible.

2. Let $U$ and $W$ be subspaces of a finite-dimensional vector space $V$.
   Show that $\dim U + \dim W = \dim(U \cap W) + \dim(U + W)$.

3. If $A$ and $B$ are real symmetric $n$-by-$n$ matrices with all eigenvalues positive, show that $A+B$ has the same property.

4. Show that any two commuting matrices with complex entries share a common eigenvector.

5. Find a cubic polynomial $P(x)$ with integer coefficients such that $P(2\cos(40°)) = 0$, and then find the Galois group of $P(x)$.

6. Show that every group of order 15 is cyclic.

7. The ring $R$ of Gaussian integers consists of all complex numbers of the form $a + bi$, where $a$ and $b$ are integers. Find a factorization of 143 into a product of 3 primes in $R$.

8. Let $K = \mathbb{Q}(x)$, the field of rational functions in one variable $x$. Let $k = \mathbb{Q}(y)$, the field of rational functions of $y$, where $y = x^2(1 + x)^2/(1 + x + x^2)^3$, so that $k$ is a subfield of $K$. Let $W$ be the group of automorphisms of $K$ generated by $\alpha$ and $\beta$, where $\alpha(x) = 1/x$ and $\beta(x) = -1 - x$. Show that an element of $K$ which is fixed by every element of $W$ must be an element of $k$.