UBC Math Qualifying Exam January 9th 2010: Morning Exam, 9am-12pm
PURE and APPLIED exam

Solve all six problems, and start each problem on a new page.

1. Prove or disprove: The series
$$\sum_{n=1}^{\infty} \frac{\sin x}{1+n^2x^2}$$
converges uniformly on $[-\pi, \pi]$.

2. Let $Q = \{0 < x < 1,\ 0 < y < 1\}$. For what values of a, b is the function
$$x^a y^b \int_0^{\infty} \frac{1}{(x+t)(y^2+t^2)}\, dt$$
bounded on Q?

3. (a) Does $p_N = \prod_{n=2}^{N}(1 + \frac{(-1)^n}{n})$ converge to a nonzero limit as $N \to \infty$? Explain your answer!
   (b) Prove that $\int_0^{\infty} \cos(t^2)dt$ converges.

4. Let $f(z) = \int_0^{\infty} e^{-zt^2} dt$.
   (a) Show that $f(z)$ is analytic in the domain $\mathrm{Re}(z) > 0$.
   (b) Assume that $f(1) = \frac{1}{2}\sqrt{\pi}$. Find the analytic continuation of $f(z)$ into the domain $\mathbb{C} \sim (-\infty, 0]$. This is the domain that is the whole complex plane except the negative real axis and zero. [Hint: compute $f(x)$ for $x > 0$.]
   (c) Let $F(z)$ denote the analytic continuation of $f(z)$ from part (b). Evaluate $F(i)$.
   (d) Evaluate $\int_0^{\infty} \cos(t^2)dt$.

5. Calculate the following integrals:
   (a)
$$\int_C (\bar{z})^2\, dz$$
   where C is the circle $|z+1| = 4$, oriented counterclockwise.
   (b)
$$\int_C z\sin(z^{-1})dz$$
   where C is the circle $|z| = 100$, oriented counterclockwise.
   (c)
$$\int_C \frac{\sin 3z}{(z-1)^4} dz$$
   where C is the circle $|z| = 2$, oriented counterclockwise.

6.
$$J = \int_0^{\infty} \frac{(\ln x)^2}{x^2+9}\, dx.$$
Evaluate J, explaining all steps and calculations carefully.

UBC Math Qualifying Exam January 9th 2010: Afternoon Exam, 1pm-4pm
APPLIED exam

Solve all seven problems, and start each problem on a new page.

1. Let $\mathbf{x}$ be a unit vector in $\mathbb{R}^n$ and let $A = I - \beta \mathbf{x}\mathbf{x}^T$.

   (a) Show that A is symmetric.

   (b) Find all values of β for which A is orthogonal.

   (c) Find all values of β for which A is invertible.

2. Let U and W be subspaces of a finite-dimensional vector space V.
   Show that $\dim U + \dim W = \dim(U \cap W) + \dim(U + W)$.

3. If A and B are real symmetric n-by-n matrices with all eigenvalues positive, show that $A+B$ has the same property.

4. Show that any two commuting matrices with complex entries share a common eigenvector.

5. Consider the ODE given by $x^2(1-x^2)y'' + 2x(1-x)y' - y = 0$.

   (a) Find and classify the finite singular points of the ODE.

   (b) Find the *form* of a general series solution of the ODE about $x = \frac{1}{2}$. (*Do not evaluate* the coefficients but do indicate which are arbitrary.)

   (c) For which values of x would the series solution of (b) converge absolutely?

   (d) Find the *form* of a general series solution of the ODE valid near $x = 0$. (*Do not evaluate* the coefficients but do indicate which are arbitrary.)

   (e) For which values of x would the series solution of (d) converge absolutely?

6. Consider the Frenet-Serret formulas for a space curve $\mathbf{r}(s)$ given by the system of differential equations

$$\begin{aligned}\frac{d\mathbf{T}}{ds} &= \kappa(s)\mathbf{N} \\ \frac{d\mathbf{N}}{ds} &= -\kappa(s)\mathbf{T} + \tau(\mathbf{s})\mathbf{B} \\ \frac{d\mathbf{B}}{ds} &= -\tau(s)\mathbf{N}\end{aligned}$$

$\mathbf{T} = d\mathbf{r}/ds$ is the tangent to the curve and $\mathbf{N}$ and $\mathbf{B}$ are unit vectors. Show that $\mathbf{r}(s)$ lies on a circular path if and only if $\kappa(s) = \text{constant} = K$, $\tau(s) = 0$. Find the radius of the circular path.

7. Suppose $K(x, \xi, t, \tau)$ satisfies

$$\begin{aligned}K_t - K_{xx} &= \delta(x-\xi)\delta(t-\tau), \qquad 0 < x, \xi < 1, \qquad t > 0 \\ K(x, \xi, 0, \tau) &= 0 \\ K_x(0, \xi, t, \tau) &= K_x(1, \xi, t, \tau) = 0.\end{aligned}$$

   (a) Find $K(x, \xi, t, \tau)$.

   (b) In terms of $K(x, \xi, t, \tau)$ and given data $\{F(x,t), f(x), h(t), k(t)\}$, find the solution $u(x,t)$ of the boundary value problem given by

$$\begin{aligned}u_t - u_{xx} &= F(x,t), \qquad 0 < x < 1, \qquad t > 0 \\ u(x, 0) &= f(x), \qquad 0 \le x \le 1 \\ u_x(0,t) &= h(t), \qquad u_x(1,t) = k(t), \qquad t > 0\end{aligned}$$

   (c) In terms of $K(x, \xi, t, \tau)$ found in (a), is your solution $u(x,t)$ useful for large or small times t? Explain your answer.