Winter 2008, Applied Qualifying Exam

Part 1

1. Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of non-negative real numbers and that $\sum_{n=1}^{\infty} a_n$ converges. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.

2. The periodic function $f(x)$ is defined by

$$f(x) = e^x, \quad \text{for } -\pi \leq x \leq \pi \text{ and } f(x+2\pi) = f(x).$$

Find the Fourier series representation of $f(x)$. Check whether the series can be differentiated to give the familiar result,

$$\frac{d}{dx}(e^x) = e^x.$$

Relate your results to the continuity properties of $f(x)$.

3. (a) State the divergence theorem, and use it to evaluate the integral,

$$\int\int_S \mathbf{u} \cdot \mathbf{n}\, ds$$

where

$$\mathbf{u} = (xz^2, \sin x, y)$$

and S is the closed surface of the cylinder bounded by

$$x^2 + y^2 = 1, \ z = 0, \ z = 2.$$

What would the result have been had $\mathbf{u} = \nabla \times \mathbf{a}$ for any differentiable vector field $\mathbf{a}(\mathbf{x})$?

(b) State Stokes' theorem, and use it to evaluate the integral

$$\int_C \mathbf{u} \cdot \mathbf{dr}$$

where C is the unit circle $x^2 + y^2 = 1$, directed in an an anticlockwise sense, and

$$\mathbf{u} = (\cos x, 2x + y \sin y, x)$$

What would the result have been had $\mathbf{u} = \nabla \phi$ for any differentiable scalar field, $\phi(\mathbf{x})$?

4. *(i)* Prove that an orthogonal set of vectors $\{\mathbf{u_1}, \mathbf{u_2}, ..., \mathbf{u_n}\}$ in an $n-$dimensional Euclidean space is linearly independent.

(ii) Let V be a subspace of $\Re^4$ spanned by the vectors,

$$\mathbf{u_1} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{u_3} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}.$$

Using the Gram-Schmidt procedure, construct an orthogonal basis for V.

(iii) Consider the vector space formed by all polynomials, $P_n(x)$ with $-1 \le x \le 1$, of degree less than or equal to n. Consider the inner product,

$$\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x)q(x)dx.$$

Determine the quadratic polynomial, $P_2(x)$, which is normalized so that $P_2(0) = 1$ and is orthogonal to both polynomials $P_0(x) = 1$ and $P_1(x) = x$.

5. (i) Define a Hermitian matrix, and prove that all of its eigenvalues are real, and that the eigenvectors corresponding to distinct eigenvalues are orthognal.

(ii) Find a matrix P such that $P^{-1}AP$ is diagonal, where

$$A = \begin{pmatrix} 3 & 3 & 2 \\ 2 & 4 & 2 \\ -1 & -3 & 0 \end{pmatrix}.$$

6. Let A be an $n \times m$ matrix. Prove that the equation $Ax = b$ has a solution if and only if $\langle b, v \rangle = 0$ for all v in the nullspace of A^*.

Part 2

1. Using contour integration, find the definite integrals

$$(a) \qquad \int_0^\infty \frac{\ln x}{x^2+1}dx$$

and

$$(b) \qquad \int_{-\infty}^\infty \frac{e^{ikx}}{\cosh x}dx$$

with k a parameter (Hint: for (b) use a rectangular contour).

2. Find all possible Laurent expansions of

$$\frac{1}{(2+z)(z^2+1)}$$

about $z=0$.

3. Consider the annular region, D, given by $\frac{1}{5} \leq |z| \leq 1$.

(i) Show that

$$\phi(z) = 2 + \frac{\ln|z|}{\ln 5}$$

is harmonic in D.

(ii) Show that

$$w = f(z) = \frac{3z+1}{3+z}$$

is a conformal mapping of D. Show that the image, E, of D in the $w-$plane is bounded by two non-concentric circles, C_1 and C_2, with C_1 contained inside C_2.

(iii) Suppose that $\Phi(w)$ is harmonic on E such that $\Phi = 2$ on C_2 and $\Phi = 1$ on C_1. Find $\Phi(w)$.

4. Define the Wronskian, $W(x)$, of the differential equation,

$$(1-x^2)u'' - 2xu' + n(n+1)u = 0,$$

where n is an integer. Find a linear first order differential equation for $W(x)$ and solve it subject to the initial condition, $W(0) = 1$. Establish the recurrence relation between the coefficients of the series solution,

$$u = \sum_{m=0} a_m x^m,$$

and hence show that there are regular polynomial solutions. For $n=1$, find such a solution explicitly if $u(1) = 1$. Use this solution to find another, independent solution.

5. Consider the PDE,

$$u_t = (x^2 u_x)_x, \qquad 1 \leq x \leq 3, \qquad u(1,t) = u(3,t) = 0, \qquad u(x,0) = f(x).$$

Show that the solution can be written in terms of a sum over the eigenfunctions of a related Sturm-Liouville problem, $\phi_n(x)$, where

$$\phi_n(x) = \frac{1}{\sqrt{x}} \sin\left(\frac{n\pi \ln x}{\ln 3}\right).$$

6. An age-structure model of a population is based on the PDE,

$$h_t + h_a = -\mu(a)h,$$

where $h(a,t)da$ gives the number of individuals with ages in the range $[a, a+da]$ at time t. The death rate, $\mu(a)$, and initial population, $h(a,0) = H(a)$, are prescribed functions. The population is sterile, so there are no births: $h(0,t) = 0$.

(a) Find a general solution using the Laplace transform in time; the survival function,

$$S(a) = \exp\left[-\int_0^a \mu(a')da'\right],$$

should feature in your solution.

(b) Solve the equation using the method of characteristics.