Spring 2007 Pure Math Qualifying Exam, Part 1.

1. If $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, how many matrices commute with A and have eigenvalues 2,3,4?

2. Assume that a and b are positive real numbers. Evaluate the following integral:
$$\int_0^\infty \frac{\sin(ax)}{x(x^2+b^2)}dx.$$

3. If f is continuous on $[0,1]$ and $f(0) = 1$, for what real numbers $r \geq 0$ does the limit
$$\lim_{n\to\infty} n^r \int_0^1 e^{-nt} f(t)\, dt$$
exist, and what is its value when it does?

4. Show that $\cos\frac{\pi}{12} \in \mathbb{Q}[\sqrt{2}+\sqrt{3}]$.

5. Let U and V be domains in $\mathbb{C}$. The Beltrami differential of a diffeomorphism $\phi : U \to V$ is $\mu(\phi) = \phi_{\bar{z}}/\phi_z = \frac{\phi_x + i\phi_y}{\phi_x - i\phi_y}$. Define a relation on diffeomorphisms from U to V by: $\phi_1 \sim \phi_2$ iff $\phi_1^{-1}\phi_2$ is complex analytic. Show that this is an equivalence relation. Show that $\phi_1 \sim \phi_2$ iff $\mu(\phi_1) = \mu(\phi_2)$.

6. Determine all groups of order 21 up to isomorphism.

Spring 2007 Pure Math Qualifying Exam, Part 2.

1. What is the radius of convergence of the power series for $\sqrt{2-e^z}$ around $z = 1 + 4i$?

2. Find the maximum value of
$$\frac{x(1-x)(1-y)}{1-xy}$$
in the domain $(x, y) \in [0, 1]^2$. Give the values of x, y where the maximum is achieved.

3. Prove Stirling's approximation $n! = n^n e^{-n}\sqrt{n}e^{O(1)}$. That is, show that the ratio of $n!$ and $n^n e^{-n} n^{1/2}$ is bounded between two positive constants, for n big enough. (Hint: one way is to take logs of both sides and use an integral approximation for $\log n!$).

4. Find all integers n such that $x^5 - nx - 1$ is irreducible over $\mathbb{Q}$.

5. Let G be the abelian group defined by generators x, y, and z, and relations
$$\begin{aligned} 15x + 3y &= 0 \\ 3x + 7y + 4z &= 0 \\ 18x + 14y + 8z &= 0. \end{aligned}$$
   (a) Express G as a direct product of two cyclic groups.
   (b) Express G as a direct product of cyclic groups of prime power order.
   (c) How many elements of G have order 2?

6. Show that for each $n = 1, 2, \ldots$ there is a unique polynomial $P_n(x)$ of degree n satisfying
$$\int_x^{x+1} P_n(t)dt = x^n$$
for all x. Compute $P_1(x)$ and $P_2(x)$.