Spring 2007 Applied Math Qualifying Exam, Part 1.

1. If $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, how many matrices commute with A and have eigenvalues 2,3,4?

2. Assume that a and b are positive real numbers. Evaluate the following integral:
$$\int_0^\infty \frac{\sin(ax)}{x(x^2+b^2)}dx.$$

3. If f is continuous on $[0,1]$ and $f(0)=1$, for what real numbers $r \geq 0$ does the limit
$$\lim_{n\to\infty} n^r \int_0^1 e^{-nt} f(t)\,dt$$
exist, and what is its value when it does?

4. Consider the partial differential equation:
$$u_{xx} + \frac{2}{x}u_x + u_{yy} + \lambda u = 0$$
defined on the region $0 \leq x \leq a$ and $0 \leq y \leq b$ along with the boundary conditions:
$$u = 0 \text{ on } y = 0 \text{ and } y = b$$
$$u \text{ bounded as } x \to 0 \text{ and } u_x = 0 \text{ on } x = a$$
Find the eigenfunctions and eigenvalues associated with this boundary value problem.

   Hint: It may be useful to make the substitution $v = xu$.

5. Find a harmonic function on the upper half plane which is 1 on the interval $[-1,1]$, zero for $z \in \mathbb{R} \setminus [-1,1]$, and tends to zero at ∞.

6. Let f and g be functions that are 2π-periodic in θ and consider the following problem describing the heat flux through the boundary of a circular disk of radius a:
$$u_t = \Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$
$$u_r(a,\theta,t) = f(\theta) \text{ and } u \text{ bounded as } r \to 0$$
$$u(r,\theta,0) = g(r,\theta) \qquad (1)$$
   (a) Determine a condition for the steady state solution to exist.
   (b) Subject to the condition in (a) determine a formula for the steady state solution up to an arbitrary constant.
   (c) Using (1) determine the unknown constant in the steady solution.

Spring 2007 Applied Math Qualifying Exam, Part 2.

1. What is the radius of convergence of the power series for $\sqrt{2-e^z}$ around $z = 1+4i$?

2. Find the maximum value of

$$\frac{x(1-x)(1-y)}{1-xy}$$

in the domain $(x, y) \in [0, 1]^2$. Give the values of x, y where the maximum is achieved.

3. Prove Stirling's approximation $n! = n^n e^{-n}\sqrt{n}e^{O(1)}$. That is, show that the ratio of $n!$ and $n^n e^{-n} n^{1/2}$ is bounded between two positive constants, for n big enough. (Hint: one way is to take logs of both sides and use an integral approximation for $\log n!$).

4. Assume that c and c_0 are constants that satisfy the condition $0 < c_0 < c$. Determine the solution of the following initial boundary value problem for the wave equation

$$u_{tt} = c^2 u_{xx} \text{ in } c_0 t < x < \infty,\ t > 0$$

subject to

$$u(x, 0) = f(x),\ u_t(x, 0) = 0 \text{ on } x \geq 0$$
$$u(c_0 t, t) = h(t),\ t \geq 0$$

5. Consider the matrix

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

   (a) Find a basis for the nullspace of A.

   (b) Find the eigenvectors and eigenvalues of A.

   (c) Show that A is semipositive definite, that is, $x^t Ax \geq 0$ for all vectors x.

   (d) Write $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ as a linear combination of eigenvectors of A.

6. Show that for each $n = 1, 2, \ldots$ there is a unique polynomial $P_n(x)$ of degree n satisfying

$$\int_x^{x+1} P_n(t)dt = x^n$$

for all x. Compute $P_1(x)$ and $P_2(x)$.