Pure Mathematics Qualifying Exam
January 7, 2006

Part I

PROBLEM 1. Find the critical points of

$$f(x, y) = x^2 + 2xy + 2y^2 - \frac{1}{2}y^4$$

and classify each one as a local minimum, local maximum, or saddle point.

PROBLEM 2. Let G be a finite group and $H < G$ a subgroup such that the index $[G : H] = p$ is the smallest prime number dividing the order of G. Prove that H is normal in G. (Hint: study the action of G on the cosets of H and the resulting homomorphism to the permutation group.)

PROBLEM 3. Evaluate using the method of residues:

$$\int_0^\infty \frac{x^2 dx}{x^4 + 5x^2 + 6}.$$

PROBLEM 4. Let V be the vector space of polynomials in one variable of degree at most n, with real coefficients. Given distinct real numbers $a_0, a_1, \ldots, a_n$, show that any polynomial $f(x) \in V$ can be expressed in the form

$$f(x) = c_0(x + a_0)^n + c_1(x + a_1)^n + \ldots + c_n(x + a_n)^n$$

for some $c_i \in \mathbb{R}$.

PROBLEM 5. Consider the complex multi-valued function

$$f(z) = (z^3 + z^2 - 6z)^{1/2}.$$

(a) Find a set of branch cuts of the complex plane such that on the complement of these cuts $f(z)$ can be defined as a single-valued function. Moreover, the cuts should be such that if we require $f(-1) = -\sqrt{6}$, then such a single-valued $f(z)$ is unique.

(b) Using the branch cuts from your answer to part (a), choose any point p lying on one of the cuts, but not a branch point itself. Describe the limiting behavior of $f(z)$ as z approaches p along different paths.

PROBLEM 6. Let (X, d) be a complete metric space (i.e., all Cauchy sequences converge) and let $L : X \to X$ be such that for some $k < 1$

$$d(Lx, Ly) < kd(x, y) \qquad \text{for all } x, y \in X.$$

Prove that there exists a point $z \in X$ such that $L(z) = z$ and that this z is unique.

Pure Mathematics Qualifying Exam
January 7, 2006

Part II

Problem 1. Let S be the hemisphere $\{x^2 + y^2 + z^2 = 1, z \geq 0\}$ oriented with $\mathbf{N}$ pointing away from the origin. Use the divergence theorem to evaluate the flux integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

where

$$\mathbf{F} = \big(x + \cos(z^2)\big)\mathbf{i} + \big(y + \ln(x^2 + z^5)\big)\mathbf{j} + \sqrt{x^2 + y^2}\,\mathbf{k}.$$

Problem 2. Let R be a principal ideal domain and $I \subset R$ a non-zero ideal.

(a) Give a definition of *principal ideal domain* (assume that we know what a domain is). Explain the relationship between principal ideal domains and unique factorization domains. No proofs are necessary.

(b) Prove that there are only finitely many ideals J in R that contain I.

Problem 3. Give examples of sequences of functions $f_n : \mathbb{R} \to \mathbb{R}$ satisfying the indicated convergence conditions. In each case write down the function $f_n(x)$ and also the limit $f(x)$. If your functions are simple enough (for example, piecewise linear), you may simply draw their graphs, carefully indicating relevant coordinates, rather than writing down definitions. No proofs are necessary.

(a) $f_n \to f$ pointwise, but not uniformly or in the L^2 norm.

(b) $f_n \to f$ in the L^2 norm, but not pointwise or uniformly.

(c) $f_n \to f$ pointwise, all of the f_n's continuous, but f not continuous.

(d) $f_n \to f$ pointwise, all of the f_n's integrable, but f not integrable.

(e) $f_n \to f$ uniformly, all of the f_n's and f integrable, but

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, dx \neq \int_{-\infty}^{\infty} f(x)\, dx.$$

(continued on back)

PROBLEM 4. All matrices in this problem have real coefficients and size $n \times n$.

It is well-known that a positive definite symmetric matrix A has a square root Q in the sense that

$$A = QQ^T.$$

Use this fact to show that if A and B are positive definite symmetric matrices then the eigenvalues of AB are real and positive.

PROBLEM 5. Let L be the intersection of the disks $|z| < 1$ and $|z-1| < 1$ on the complex plane. Let $f : L \to H$ be a one-to-one analytic mapping from this lens-shaped region onto the upper half-plane H.

(a) Explain why f can not be a Möbius transformation

$$f(z) = \frac{az+b}{cz+d} \qquad \text{for some } a, b, c, d \in \mathbb{C}.$$

(b) Find an f as a composition of a Möbius transformation and a power map $z \mapsto z^\alpha$ for an appropriate α.

PROBLEM 6. Factor the polynomial $x^3 - 3x + 3$ and find the Galois group of its splitting field if the ground field is

(a) $\mathbb{R}$. (You don't need to find the exact value of the root(s).)

(b) $\mathbb{Q}$.