Applied Math Qualifying Exam: Jan. 8, 2005

Part I

1. Let b and c be real numbers, $c > 0$. Use contour integration to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos(x-b)}{x^2+c^2} dx.$$

2. Let A be an $n \times n$ real symmetric matrix with smallest eigenvalue λ_1 and largest eigenvalue λ_n. Show that for any vector $v \neq 0$ in $\mathbb{R}^n$,

$$\lambda_1 \leq \frac{\langle v, Av \rangle}{\langle v, v \rangle} \leq \lambda_n$$

(here $\langle v, w \rangle = v^T w$ is the standard inner-product on $\mathbb{R}^n$).

3. (a) Find the eigenvalues λ and the eigenfunctions for the eigenvalue problem

$$u_{xx} + u_{yy} = \lambda u, \quad (x,y) \in (0,1) \times (0,1)$$

$$u(x,0) = u_x(0,y) = u_y(x,1) = u(1,y) = 0. \qquad (*)$$

(b) Now suppose $w(x,y,t)$ solves the heat equation

$$w_t = w_{xx} + w_{yy}, \quad (x,y) \in (0,1) \times (0,1), \quad t > 0$$

with the same boundary conditions as $(*)$, and with initial condition $w(x,y,0) \equiv 1$. Find the leading-order behaviour of w as $t \to \infty$.

4. Consider the vector field

$$\mathbf{F}(x,y,z) = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{3/2}}.$$

(a) Verify that $\nabla \cdot \mathbf{F} = 0$ on $\mathbb{R}^3 \backslash \{0\}$.

(b) Let S be a sphere centred at the origin, with "outward" orientation. Show that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 4\pi. \qquad (*)$$

(c) Now let $E \subset \mathbb{R}^3$ be an open region with smooth boundary S (given "outward" orientation), and suppose $0 \in E$. Show that $(*)$ still holds.

5. In the complex plane, let C_1 be the circle passing through -2, $-i$, and 2, and let C_2 be the circle passing through -2, $(2/3)i$, and 2. Let Ω be the intersection of the open disks whose boundaries are C_1 and C_2 (so Ω is the region bounded by C_1 and C_2).

   (a) Find a transformation of the form

$$z \mapsto \frac{az+b}{cz+d}, \qquad a, b, c, d \in \mathbb{C}$$

   (i.e. a fractional linear transformation) which maps -2 to 0, $-i$ to 1, and 2 to ∞.

   (b) Find the image of Ω under this mapping.

   (c) Find the angle between the circles C_1 and C_2 at the point -2.

6. Let V be the vector space of all polynomials $p(x)$ with real coefficients. Let A and B denote the linear transformations on V of (respectively) multiplication by x, and differentiation. That is, $A : p(x) \mapsto xp(x)$, and $B : p(x) \mapsto p'(x)$.

   (a) Show that A has no eigenvalues, and that 0 is the only eigenvalue of B.

   (b) Compute the transformation $BA - AB$.

   (c) Show that no two linear transformations A, B on a *finite dimensional* real vector space can satisfy $BA - AB = I$ (here I denotes the identity transformation).

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Part II

1. (a) Show that a continuous function on $\mathbb{R}$ cannot take every real value exactly twice.

   (b) Find a continuous function on $\mathbb{R}$ that takes each real value exactly 3 times.

2. A nonlinear oscillator is described by the following ODE for $y(t)$:

$$y'' + \gamma(y)y' + g(y) = 0 \qquad (*)$$

   where $\gamma(y)$ and $g(y)$ are smooth functions, with $g(0) = 0$.

   (a) Verify that the constant function $y_0(t) \equiv 0$ is a solution of $(*)$.

   (b) Re-write $(*)$ as a first-order system.

   (c) What conditions on γ and g ensure that the constant solution y_0 is stable? Unstable?

3. Show that an entire function $f(z)$ satisfying $\lim_{|z|\to\infty} |f(z)| = c$ (for some $c \in (0, \infty)$) is constant.

4. For a vector $v = (v_1, \ldots, v_n)^T \in \mathbb{R}^n$, define $\|v\|_1 := \sum_{j=1}^n |v_j|$, and for an $n \times n$ matrix A, define

$$\|A\|_1 := \sup_{v \in \mathbb{R}^n; v \neq 0} \frac{\|Av\|_1}{\|v\|_1}.$$

Show that if $A = (a_{ij})$, then

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|.$$

5. Let $f(x)$ be a periodic function with period 1 whose Fourier series is $\sum_{n=-\infty}^{\infty} a_n e^{2\pi inx}$.

(a) Find an integral formula for the coefficients a_n (in terms of f).

(b) Show that if f has m continuous derivatives, then $|a_n| \leq C/|n|^m$ (where C is a constant depending on f).

6. Consider the following PDE for $u(x,t)$:

$$u_t + auu_x + bu_{xxx} = 0, \quad -\infty < x < \infty, \quad t > 0$$

($a, b > 0$ are constants).

(a) Use scaling to reduce the problem to the form

$$w_t + ww_x + w_{xxx} = 0, \quad -\infty < x < \infty, \quad t > 0. \qquad (*)$$

(b) Suppose $w(x,t)$ is a smooth solution of $(*)$ for which w and its derivatives decay rapidly to 0 as $x \to \pm\infty$. Show that the quantity

$$\int_{-\infty}^{\infty} w^2(x,t)dx.$$

is constant in time.

(c) Equation $(*)$ has solutions of the form $w(x,t) = \phi(x - ct)$ (c a constant) with $\phi > 0$, and $\phi(y)$ (and its derivatives) tending to 0 as $y \to \pm\infty$. Find the equation satisfied by the function ϕ, and solve it. (This last part is somewhat involved!).