Pure Mathematics Qualifying Exam, January10, 2004. Part I

1. Find the shortest distance from a point on the ellipse
$$\frac{x^2}{4} + y^2 = 1$$
to the straight line $x + y = 4$.

2. Let A be an $n \times n$ matrix with complex entries. An $n \times n$-matrix B is called a *square root* of A if $B^2 = A$. Suppose A is non-singular and has n distinct eigenvalues. How many square roots does A have?

3. Let $f(z)$ be an analytic function and $|f(z)| \leq 1$ in the unit disc $D \subset \mathbb{C}$. Given $z_0 \in D$, find a Möbius transformation (i.e., a transformation of the form $z \mapsto \frac{az+b}{cz+d}$) which maps D to D and sends z_0 to 0. Then show that
$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| \leq \frac{2}{1 - |z_0||z|}$$
for any $z \in D$.

4. A rational function $f(x_1, \ldots, x_n)$ in n variables is a ratio of two polynomials,
$$f = \frac{p(x_1, \ldots, x_n)}{q(x_1, \ldots, x_n)},$$
where q is not identically 0. We shall assume throughout that the coefficients of our polynomials are real numbers. A rational function $f(x_1, \ldots, x_n)$ is called *symmetric* if $f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for any permutation σ of $\{1, \ldots, n\}$. We shall denote the field of rational functions in n variables by F and the subfield of symmetric rational functions by $S \subset F$.

   (a) Show that F is a finite extension of S of degree $n!$.

   (b) Show that $F = S(h)$, where $h = x_1 + 2x_2 + \cdots + nx_n$. In other words, show that h generates F as a field extension of F.

5. Let f be a real function on $[0, 1]$ having the following property: for any real y, the equation $f(x) - y = 0$ has either no roots, or exactly two roots. Prove that f is not continuous.

6. Define a sequence $x_1, x_2, \ldots$ recursively by $x_0 = c$, $x_1 = 1 - c$, and $x_{n+2} = 2.5x_{n+1} - 1.5x_n$ for $n \geq 1$. For what values of c does the sequence $\{x_n\}$ converge? If it converges, what is the value of $\lim_{n \to \infty} x_n$?

Pure Mathematics Qualifying Exam, January 10, 2004. Part II

7. Evaluate the integral
$$I = \int_0^\infty \frac{\cos(x)}{x^2+9}\, dx\,.$$

8. Let G be a group of order ab, where a and b are relatively prime positive integers. Suppose H is a normal subgroup of order a. Show that H contains every subgroup of G whose order divides a.

9. Let $\{f_n\}$ be an equicontinuous sequence of functions on a compact set K, which converges pointwise to a function f.

   (a) Prove that f is continuous.

   (b) Prove that $\{f_n\}$ converges uniformly to f.

10. Are the following statements true? In each case give a proof of a counterexample. Assume that A and B are $n \times n$-matrices with real entries and $n \geq 2$.

    (a) If $\det(A) = \det(B) = 1$ then $A + B$ is non-singular.

    (b) If A and B are symmetric matrices all of whose eigenvalues are strictly positive, then $A + B$ is non-singular.

11. Suppose that c is an isolated singularity of an analytic function f on $\mathbb{C}\backslash\{c\}$ and that $g(z) = e^{f(z)}$.

    (a) Show that if $g(z)$ has a pole of order m at $z = c$, then $f'(z)$ has a simple pole of residue $-m$ at $z = c$.

    (b) Use this to show that $g(z)$ must have an essential singularity at $z = c$.

12. Prove that every finite multiplicative subgroup of the complex numbers is cyclic.