Algebra Qualifying Exam

University of British Columbia

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1. Consider the system of linear equations with real coefficients

$$\begin{aligned} x_1 \qquad\quad + x_3 - x_4 &= -4 \\ x_1 + 2x_2 - x_3 + 3x_4 &= 2 \\ 2x_1 + 4x_2 - 2x_3 + 7x_4 &= 5 \\ x_2 - x_3 + 2x_4 &= 3 \end{aligned}$$

   (a) Find all solutions to this system of equations.

   (b) The system can be written in the matrix form as $A\vec{x} = \vec{b}$. Let $L_A : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation defined by the matrix A. Find a basis for the kernel and a basis for the image of L_A.

2. Let P_3 be the vector space of polynomials in one variable with real coefficients and of degree at most 3. Let $T : P_3 \to P_3$ be the linear operator

$$T(f(x)) = xf''(x) + 2f(x).$$

   (a) Find the matrix of T with respect to some basis of P_3.

   (b) Find the Jordan canonical form and a Jordan canonical basis for T.

3. Let $T : V \to V$ be a linear operator on a finite dimensional vector space V. Let $W \subset V$ be a subspace, such that $T(W) \subset W$.

   (a) Assume that $\vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_n \in W$ for some vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$ that are eigenvectors of T corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Prove that then the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ lie in W.

   (b) Let $T|_W : W \to W$ be the restriction of T to W. Prove that if T is diagonalizable, then $T|_W$ is also diagonalizable.

4. (a) Give an explicit description of one Sylow 3-subgroup in S_6 and find the number of all Sylow 3-subgroups in S_6.

   (b) Let H be the subgroup of S_7 generated by $\sigma = (1\,2\,3\,4\,5\,6\,7)$ and $\tau = (2\,3\,5)(4\,7\,6)$. Prove that H is a non-abelian group of order 21.

   (c) Show that there are as many non-isomorphic finite abelian groups of order 2^{36} as the number of conjugacy classes in the symmetric group S_{36}.

5. (a) Which of the rings $\mathbb{Q}[x]/(x^4 + 1)$, $\mathbb{R}[x]/(x^4 + 1)$ is a field. Justify your answer with full details.

(b) Let $S = \{f \in \mathbb{R}[x] \mid f(2) = f'(2) = f''(2) = 0\}$. Show that S is an ideal of $\mathbb{R}[x]$ and give a generator of S.

(c) Consider the ring $R = \mathbb{Q}[x]/(f(x))$, where $f(x) \in \mathbb{Q}[x]$ is a nonconstant polynomial. Prove that the intersection of all maximal ideals of R is equal to the set of all nilpotents in R. Your proof must include an explicit description of all maximal ideals and all nilpotents in R. (Recall that $r \in R$ is nilpotent if $r^n = 0$ for some $n > 0$.)

6. Let p be an odd prime and $\mathbb{F}_p$ the finite field containing p elements. Let $\mathrm{GL}_2(\mathbb{F}_p)$ be the group of 2×2 matrices over the field $\mathbb{F}_p$ with non-zero determinant (the group operation is matrix multiplication). Consider two subgroups, $\mathrm{SL}_2(\mathbb{F}_p)$ the set of matrices of determinant 1 and U the set of upper triangular matrices

$$U = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a,\, d \in \mathbb{F}_p^{\times}, b \in \mathbb{F}_p \right\}.$$

Suppose that L/K is a Galois extension with Galois group $\mathrm{Gal(L/K)} \simeq \mathrm{GL}_2(\mathbb{F}_p)$. Let L_1 be the fixed field of $\mathrm{SL}_2(\mathbb{F}_p)$ and let L_2 be the fixed field of U.

(a) Compute the degrees $[L : K]$, $[L_1 : K]$ and $[L_2 : K]$.

(b) What is the Galois group $\mathrm{Gal}(L/L_1L_2)$?

(c) Show that L_1L_2 is not a Galois extension of K, but is Galois over L_2 and compute $\mathrm{Gal}(L_1L_2/L_2)$.