Differential Equations Qualifying Exam

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1. Consider the system of linear equations with real coefficients

$$\begin{aligned} x_1 \qquad\quad +x_3 - x_4 &= -4 \\ x_1 + 2x_2 - x_3 + 3x_4 &= 2 \\ 2x_1 + 4x_2 - 2x_3 + 7x_4 &= 5 \\ x_2 - x_3 + 2x_4 &= 3 \end{aligned}$$

   (a) Find all solutions to this system of equations.

   (b) The system can be written in the matrix form as $A\vec{x} = \vec{b}$. Let $L_A : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation defined by the matrix A. Find a basis for the kernel and a basis for the image of L_A.

2. Let P_3 be the vector space of polynomials in one variable with real coefficients and of degree at most 3. Let $T : P_3 \to P_3$ be the linear operator

$$T(f(x)) = xf''(x) + 2f(x).$$

   (a) Find the matrix of T with respect to some basis of P_3.

   (b) Find the Jordan canonical form and a Jordan canonical basis for T.

3. Let $T : V \to V$ be a linear operator on a finite dimensional vector space V. Let $W \subset V$ be a subspace, such that $T(W) \subset W$.

   (a) Assume that $\vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_n \in W$ for some vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$ that are eigenvectors of T corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Prove that then the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ lie in W.

   (b) Let $T|_W : W \to W$ be the restriction of T to W. Prove that if T is diagonalizable, then $T|_W$ is also diagonalizable.

4. Consider the third order differential equation for $y(t)$ for $t \geq 0$ given by

$$y''' - 4y' = -4 + e^{-t}, \quad t \geq 0$$

   (a) Determine the explicit form of the general solution to this problem.

   (b) For the initial values $y(0) = 0$, $y'(0) = \alpha$ and $y''(0) = 1$ find the unique value of α for which $\lim_{t\to\infty} y' = 1$.

5. Let ω, A, and T be constants with $\omega > 0$, $T > 0$. Consider the mass-spring system for $y(t)$ subject to a delta-function forcing, modeled by

$$\begin{aligned} &y'' + \omega^2 y = A\delta(t - T), \quad t \geq 0 \\ &y(0) = 1, \;\; y'(0) = -1. \end{aligned}$$

(a) Calculate the solution using Laplace transforms.
<u>Some potentially useful Laplace transforms</u>:

$f(t)$	$\mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$, $s > 0$
t	$\frac{1}{s^2}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
$\sin(at)$	$\frac{a}{s^2+a^2}$, $s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$, $s > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2}$, $s > \|a\|$
$\cosh(at)$	$\frac{s}{s^2-a^2}$, $s > \|a\|$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$

(b) Now set $\omega = 1$. From your solution in (a) show that one can choose A and T so that $y = 0$ for all $t > T$. (This shows that an appropriate delta function forcing at time $t = T$ can extinguish the oscillation that exists for $0 \le t < T$).

6. This problem concerns the wave equation $u_{tt} - u_{xx} + m^2 u = f(x,t)$ on the whole line where m is a nonnegative constant.

(a) Assume that $f = 0$ and that u is a solution which is C^2 and that $u(x,t)$ is zero for x outside a sufficiently large interval for each t. Show that the energy $E(t) = \frac{1}{2}\int_{-\infty}^{\infty}(u_t^2 + u_x^2 + m^2u^2)\,dx$ is conserved (i.e. independent of t).

(b) For this part we assume that $m = 0$ and $f = 0$. Determine explicitly the solution $Q(x,t)$ of $u_{tt} - u_{xx} = 0$ with initial conditions $u(x,0) = 0$, $u_t(x,0) = \psi(x)$ where $\psi(x) = 0$ for $x \le 0$ and $\psi(x) = 1$ for $x > 0$.

(c) Let $Q(x,t)$ be as in part (b). Calculate $S(x,t) = Q_x(x,t)$ for $t > 0$ wherever the derivative exists, and show that the solution of the inhomogeneous wave equation $u_{tt} - u_{xx} = f(x,t)$ with initial conditions $u(x,0) = u_t(x,0) = 0$ may be written

$$u(x,t) = \int_0^t \left(\int_{-\infty}^{\infty} S(x-y,t-s) f(y,s)\,dy\right) ds.$$