The University of British Columbia
Department of Mathematics
Qualifying Examination—Differential Equations
September 2021

Linear Algebra

1. (10 points) Consider a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ which produces the following output:

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}.$$

   (a) What is a matrix representation of A?

   (b) Compute the determinant of A.

   (c) Find the eigenvalues of A.

2. (5 points) Consider a dataset consisting of n vectors $x_1, \ldots, x_n \in \mathbb{R}^d$. The sample covariance matrix is defined to be $S = \frac{1}{n} \sum_{i=1}^{n} x_i x_i^T$. Find a simple expression for S in terms of the data matrix $X \in \mathbb{R}^{d \times n}$ whose columns correspond to the data vectors:

$$X = \begin{bmatrix} | & & | \\ x_1 & \ldots & x_n \\ | & & | \end{bmatrix}.$$

3. (15 points) Let $A \in \mathbb{R}^{d \times d}$ be a real, symmetric matrix. Suppose the eigenvalues are distinct and ordered in decreasing order $\lambda_1 > \lambda_2 > \lambda_3 > \ldots > \lambda_d > 0$. Denote the eigenvectors of A by $v_1, \ldots, v_d$.

   (a) Let $b \in \mathbb{R}^d$ be a vector that is not an eigenvector of A and which satisfies $\|b\| = 1$. Show that

$$\left| v_1^T \frac{Ab}{\|Ab\|} \right| > |v_1^T b|.$$

   (b) Consider the map $T : \mathbb{R}^d \to \mathbb{R}^d$ given by

$$T(b) = \frac{Ab}{\|Ab\|},$$

   where b is an arbitrary non-zero vector. What is $\lim_{n\to\infty} T^n(b)$? Justify your answer.

Differential Equations

4. (a) *[4 points]* **Bessel's equation** is

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(1 - \frac{m^2}{z^2}\right) y = 0,$$

and has the solution, $y(z) = J_m(z)$, which is regular at $z = 0$ and satisfies (for any real constant α)

$$\int_0^x x[J_m(\alpha x)]^2 dx = \frac{1}{2}x^2[J_m'(\alpha x)]^2 + \frac{1}{2}\left(x^2 - \frac{m^2}{\alpha^2}\right)[J_m(\alpha x)]^2.$$

Given that $J_0 \approx 1 - \frac{1}{2}z^2 + ...$ and $J_1 \approx z + ...$ for $z \ll 1$, differentiate Bessel's equation for $m = 0$ to establish that $J_1(z) = -J_0'(z)$. Then show that

$$\int_0^z zJ_0(z)dz = zJ_1(z) \qquad \& \qquad \int_0^z z^2 J_1(z)dz = 2zJ_1(z) - z^2 J_0(z).$$

(b) *[6 points]* Using Sturm-Liouville theory establish that the eigenvalue problem,

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \lambda y - \frac{y}{x^2} = 0,$$

has an eigenvalue λ that is positive if $y(x)$ is regular on $0 \le x \le 1$ and satisfies $y(1) = 0$. What are the eigenfunctions $y(x)$? Show that

$$x = -\sum_{n=1} \frac{2J_0(k_n)J_1(k_n x)}{k_n[J_1'(k_n)]^2},$$

for some constants k_n.

5. (10 points) (a) An ODE model for the inter-personal relationship between two individuals is as follows: a couple has attraction $x(t)$ but repulsion $y(t)$, leading to the degree of happiness $z(t)$ and anguish $w(t)$, all satisfying the ODEs,

$$x' + x + y = 0, \quad y' - y - 2x = 0, \quad z' + z = y + 1, \quad w' + w^2(x + 1) = 0.$$

If the relationship begins with pure repulsion and anguish, $(x(0), y(0), z(0), w(0)) = (0, 1, 0, 1)$, find the solution. What happens for long times? Is a long term relationship sensible?

(b) Using variation of constants or otherwise, solve

$$y'' + 2y' + y = t^a e^{-t}$$

for $y(0) = y'(0) = 0$ and $a > 0$ is a parameter.

6. (10 points) Use separation of variables to solve

$$(r^2 u_r)_r + u_{\theta\theta} + \frac{1}{4}u = 0,$$

on the semicircle, $r \le 1$ and $0 \le \theta \le \pi$, subject to $u(r, 0) = u(r, \pi) = 0$ and $u(1, \theta) = f(\theta)$. Recalling that $\sum_{n=1}^{\infty} z^n = z/(1 - z)$, sum your series for $u(r, \theta)$, and hence write down a compact expression for the solution in terms of a single integral. Evaluate the integral if the boundary function is localized with $f(\theta) = \delta(\theta - \frac{1}{2}\pi)$, where $\delta(x)$ is Dirac's delta-function.