Linear Algebra

1. (10 points) Consider a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ which produces the following output:

\[
A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix},
\]

\[
A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}.
\]

(a) What is a matrix representation of $A$?

(b) Compute the determinant of $A$.

(c) Find the eigenvalues of $A$.

2. (5 points) Consider a dataset consisting of $n$ vectors $x_1, \ldots, x_n \in \mathbb{R}^d$. The sample covariance matrix is defined to be $S = \frac{1}{n} \sum_{i=1}^{n} x_i x_i^T$. Find a simple expression for $S$ in terms of the data matrix $X \in \mathbb{R}^{d \times n}$ whose columns correspond to the data vectors:

\[
X = \begin{bmatrix} | & | \\ x_1 & \ldots & x_n \\ | & | \end{bmatrix}.
\]

3. (15 points) Let $A \in \mathbb{R}^{d \times d}$ be a real, symmetric matrix. Suppose the eigenvalues are distinct and ordered in decreasing order $\lambda_1 > \lambda_2 > \lambda_3 > \ldots > \lambda_d > 0$. Denote the eigenvectors of $A$ by $v_1, \ldots, v_d$.

(a) Let $b \in \mathbb{R}^d$ be a vector that is not an eigenvector of $A$ and which satisfies $\|b\| = 1$. Show that

\[
\left| v_1^T \frac{Ab}{\|Ab\|} \right| > \left| v_1^T b \right|.
\]

(b) Consider the map $T : \mathbb{R}^d \to \mathbb{R}^d$ given by

\[
T(b) = \frac{Ab}{\|Ab\|},
\]

where $b$ is an arbitrary non-zero vector. What is $\lim_{n \to \infty} T^n(b)$? Justify your answer.
Differential Equations

4. (a) [4 points] Bessel’s equation is
\[ \frac{d^2y}{dz^2} + \frac{1}{z} \frac{dy}{dz} + \left(1 - \frac{m^2}{z^2}\right)y = 0, \]
and has the solution, \( y(z) = J_m(z) \), which is regular at \( z = 0 \) and satisfies (for any real constant \( \alpha \))
\[ \int_0^x x J_m(\alpha x)^2 dx = \frac{1}{2} x^2 [J_m'(\alpha x)]^2 + \frac{1}{2} \left(x^2 - \frac{m^2}{\alpha^2}\right) [J_m(\alpha x)]^2. \]

Given that \( J_0 \approx 1 - \frac{1}{2} z^2 + \ldots \) and \( J_1 \approx z + \ldots \) for \( z \ll 1 \), differentiate Bessel’s equation for \( m = 0 \) to establish that \( J_1(z) = -J_0'(z) \). Then show that
\[ \int_0^z x J_0(z) dz = z J_1(z) \quad \& \quad \int_0^z x^2 J_1(z) dz = 2 z J_1(z) - z^2 J_0(z). \]

(b) [6 points] Using Sturm-Liouville theory establish that the eigenvalue problem,
\[ \frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \lambda y - y = 0, \]
has an eigenvalue \( \lambda \) that is positive if \( y(x) \) is regular on \( 0 \leq x \leq 1 \) and satisfies \( y(1) = 0 \). What are the eigenfunctions \( y(x) \)? Show that
\[ x = -\sum_{n=1}^{\infty} \frac{2 J_0(k_n) J_1(k_n x)}{k_n [J_1'(k_n)]^2}, \]
for some constants \( k_n \).

5. (10 points) (a) An ODE model for the inter-personal relationship between two individuals is as follows: a couple has attraction \( x(t) \) but repulsion \( y(t) \), leading to the degree of happiness \( z(t) \) and anguish \( w(t) \), all satisfying the ODEs,
\[ x' + x + y = 0, \quad y' - y - 2x = 0, \quad z' + z = y + 1, \quad w' + w^2(x + 1) = 0. \]

If the relationship begins with pure repulsion and anguish, \( (x(0), y(0), z(0), w(0)) = (0, 1, 0, 1) \), find the solution. What happens for long times? Is a long term relationship sensible?

(b) Using variation of constants or otherwise, solve
\[ y'' + 2y' + y = t^a e^{-t} \]
for \( y(0) = y'(0) = 0 \) and \( a > 0 \) is a parameter.

6. (10 points) Use separation of variables to solve
\[ (r^2 u_r)_r + u_{\theta \theta} + \frac{1}{4} u = 0, \]
on the semicircle, \( r \leq 1 \) and \( 0 \leq \theta \leq \pi \), subject to \( u(r, 0) = u(r, \pi) = 0 \) and \( u(1, \theta) = f(\theta) \). Recalling that \( \sum_{n=1}^{\infty} z^n = z/(1 - z) \), sum your series for \( u(r, \theta) \), and hence write down a compact expression for the solution in terms of a single integral. Evaluate the integral if the boundary function is localized with \( f(\theta) = \delta(\theta - \frac{1}{2} \pi) \), where \( \delta(x) \) is Dirac’s delta-function.