Real analysis

1. (10 points) Let \( \mathbf{F} \) be the vector field \((x + e^{y^2}) \mathbf{i} + (y - \sin(z^2)) \mathbf{j} + z^2 \mathbf{k}\), and let \( S \) be the boundary of the region \( V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 4, 0 \leq z \leq 1\} \), oriented so that the normal points outwards. Calculate the flux integral \( \iint_S \mathbf{F} \cdot \mathbf{n} \, dS \).

2. (10 points) Determine if the following assertions are true or false, justifying the answers carefully.
   (a) Let \((a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty} \) be real numbers. If the series \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) both converge, then the series \( \sum_{n=1}^{\infty} a_n b_n \) converges.
   (b) Let \((a_n)_{n=1}^{\infty} \) be real numbers such that \( \sum_{n=1}^{\infty} a_n \) converges absolutely. Then the sequence of functions \( f_N(x) = \sum_{n=1}^{N} a_{x^n} \) defined on \([-1, 1]\) converges uniformly as \( N \to \infty \).
   (c) If the sequence of continuously differentiable functions \( f_n : [0, 1] \to \mathbb{R} \) converges uniformly to \( f \), then \( f \) must be differentiable on all of \([0, 1]\).

3. (10 points) (a) (3 points) Let \( D = \{(x, y) \in \mathbb{R}^2 : xy \in \mathbb{Q}\} \). Prove that both \( D \) and \( \mathbb{R}^2 \setminus D \) are dense in \( \mathbb{R}^2 \). You may use the fact that rationals and irrationals are dense in \( \mathbb{R}^2 \).
   (b) (7 points) Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be the function defined by
      \[
      f(x, y) = \begin{cases} 
      x & \text{if } xy \in \mathbb{Q} \\
      y & \text{if } xy \notin \mathbb{Q} 
      \end{cases}
      .
      
      Find all the points \((a, b) \in \mathbb{R}^2 \) such that \( f \) is continuous at \((a, b)\).
Complex analysis

4. (10 points) Compute the contour integral

\[ \oint_C \frac{z + 1}{z^3 + 2z^2} \, dz, \]

where \( C \) denotes

(a) (5 points) the circle \( \{ z : |z| = 1 \} \) traversed once in the counterclockwise direction.

(b) (5 points) the circle \( \{ z : |z + 2 - i| = 2 \} \) traversed once in the counterclockwise direction.

5. (10 points) (a) (7 points) Find all singularities of the function

\[ f(z) = \frac{z^3}{1 - \cos(z^2)}. \]

Determine the nature of each singularity (i.e., whether it is removable, essential or a pole). For each pole, determine its order.

(b) (3 points) Find all entire functions \( f : \mathbb{C} \to \mathbb{C} \) such that \( f(0) = 3 \) and \( |f(z)| \leq 8 |e^z| \) for all \( z \in \mathbb{C} \).

6. (10 points) (a) (5 points) Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function. Let \( A, B > 0 \) be positive real numbers such that

\[ |f(z)| \leq A |z| + B \quad \text{for all } z \in \mathbb{C}. \]

Show that there exists \( a, b \in \mathbb{C} \) such that

\[ f(z) = az + b \quad \text{for all } z \in \mathbb{C}. \]

Hint: Show that \( f'' \) is identically zero.

(b) (5 points) Find the number of zeros (where each zero is counted as many times as its multiplicity) of the polynomial \( f(z) = z^6 - 5z^2 + 10 \) in the annulus \( A = \{ z : 1 < |z| < 2 \} \).

Hint: Apply Rouché’s theorem.