The University of British Columbia
Department of Mathematics
Qualifying Examination—Algebra
September 2021

Linear Algebra

1. (10 points) Consider a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ which produces the following output:

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}.$$

   (a) What is a matrix representation of A?

   (b) Compute the determinant of A.

   (c) Find the eigenvalues of A.

2. (5 points) Consider a dataset consisting of n vectors $x_1, \ldots, x_n \in \mathbb{R}^d$. The sample covariance matrix is defined to be $S = \frac{1}{n}\sum_{i=1}^{n} x_i x_i^T$. Find a simple expression for S in terms of the data matrix $X \in \mathbb{R}^{d\times n}$ whose columns correspond to the data vectors:

$$X = \begin{bmatrix} | & & | \\ x_1 & \ldots & x_n \\ | & & | \end{bmatrix}.$$

3. (15 points) Let $A \in \mathbb{R}^{d\times d}$ be a real, symmetric matrix. Suppose the eigenvalues are distinct and ordered in decreasing order $\lambda_1 > \lambda_2 > \lambda_3 > \ldots > \lambda_d > 0$. Denote the eigenvectors of A by $v_1, \ldots, v_d$.

   (a) Let $b \in \mathbb{R}^d$ be a vector that is not an eigenvector of A and which satisfies $\|b\| = 1$. Show that

$$\left| v_1^T \frac{Ab}{\|Ab\|} \right| > |v_1^T b|.$$

   (b) Consider the map $T : \mathbb{R}^d \to \mathbb{R}^d$ given by

$$T(b) = \frac{Ab}{\|Ab\|}.$$

   What is $\lim_{n\to\infty} T^n(b)$? Justify your answer.

Advanced Algebra

4. (8 points) Let R be a commutative ring with unit and let I and J be ideals.
   (a) State the definitions of the ideals IJ and $I \cap J$, and show that $IJ \subseteq I \cap J$.
   (b) Give an example, with proof, of a specific R and distinct ideals I and J such that IJ and $I \cap J$ are not equal.
   (c) If there exists an $x \in I$ and a $y \in J$ so that $x + y = 1$, prove that $I \cap J = IJ$.

5. (10 points) Let p be a prime.
   (a) Prove that a group of order p^k, for k a positive integer, has non-trivial centre.
   (b) Prove that a group of order p^2 must be abelian. Note: you may make use of part (a) even if you have not solved it.

6. (12 points) For this problem, R is a commutative ring and G is a group. You will need three definitions.

   The *group ring* RG consists of all finite formal linear combinations $\sum_{i=1}^{m} r_i g_i$ where $r_i \in R$ and $g_i \in G$, with product defined by the formula

   $$\sum_{i=1}^{m} r_i g_i \cdot \sum_{j=1}^{n} s_j h_j = \sum_{i=1}^{m} \sum_{j=1}^{n} r_i s_j g_i h_j \tag{1}$$

   (note that the group operation is written multiplicatively). Said another way, RG is the free left R-module generated by the elements of G.

   Let G be a group with identity element e. A *positive cone* for G is a subset $\mathcal{P} \in G$ satisfying:

   - **(multiplicative closure)** If $g, h \in \mathcal{P}$ then $gh \in \mathcal{P}$.
   - **(partition)** For all $g \in G$ exactly one of $g \in \mathcal{P}$ or $g^{-1} \in \mathcal{P}$ or $g = e$ holds.

   Said another way, $\mathcal{P}$ is a subsemigroup of G for which G may be partitioned as $\mathcal{P} \sqcup \{e\} \sqcup \mathcal{P}^{-1}$.

   Finally, for any ring recall that a *zero divisor* is a non-zero element r in the ring for which there exists a non-zero r' satisfying $rr' = 0$.

   (a) Let $R = \mathbb{Z}$ and let G be the cyclic group on 5 elements. By giving an explicit example and proof, show that RG has zero divisors.
   (b) Suppose G is a group admitting a positive cone $\mathcal{P}$. Show that if $g \in G$ and n is a positive integer such that $g^n = e$ then $g = e$.
   (c) Suppose that G admits a positive cone $\mathcal{P}$ and consider the group ring RG. Assume that in the product (1) the g_i are distinct and the h_j satisfy $h_j^{-1} h_{j+1} \in \mathcal{P}$ for $1 \leq j < n$. Prove that there is an element $g_k h_1$ satisfying $(g_k h_1)^{-1} g_i h_j \in \mathcal{P}$ for all $i \neq k$ and $j \neq 1$.
   (d) Suppose that G admits a positive cone and R has no zero divisors. Prove that RG has no zero divisors.