The University of British Columbia
Department of Mathematics
Qualifying Examination—Differential Equations
January 2021

Differential equations

1. (15 points) Consider the following eigenvalue problem for $\Phi(x)$ with eigenvalue parameter λ:

$$\begin{aligned} x\Phi'' - \Phi' - \Phi &= -x\lambda\Phi\,, \qquad 1 < x < 2\,, \\ \Phi(1) = 0\,, \quad \Phi'(2) &= -\Phi(2)\,. \end{aligned} \tag{1}$$

   (a) (4 points) Prove that any eigenvalue λ for (1) must be real-valued.

   (b) (4 points) Then, prove that any eigenvalue λ for (1) must satisfy $\lambda > 0$.

   (c) (3 points) State and derive the orthogonality relation for eigenfunctions of (1).

   (d) (4 points) Finally, suppose that $f(x)$ satisfies the boundary value problem

$$\begin{aligned} xf'' - f' - f &= 1\,, \qquad 1 < x < 2\,, \\ f(1) = 0\,, \quad f'(2) &= -f(2)\,. \end{aligned} \tag{2}$$

   Find a formula for the coefficients c_n in the eigenfunction representation $f(x) = \sum_{n=1}^{\infty} c_n \Phi_n(x)$ for the solution to (2). Here, $\Phi_n(x)$ for $n \geq 1$ are the eigenfunctions of (1).

2. (15 points) Let $\omega > 0$ be a real-valued constant, and consider the fourth-order initial-value problem, defined on $t \geq 0$, for $y(t)$

$$y'''' - y = 4\cos(\omega t)\,. \tag{4}$$

   (a) (5 points) For $\omega \neq 1$, find the general solution to (4) in terms of arbitrary coefficients.

   (b) (4 points) Consider (4) with $\omega \neq 1$ with the initial values $y(0) = A$ and $y'(0) = y''(0) = y'''(0) = 0$. Determine a formula for A in terms of ω so that $y(t)$ is bounded as $t \to \infty$.

   (c) (3 points) Find the particular solution to (4) when $\omega = 1$.

   (d) (3 points) Finally, for $\omega \neq 1$ consider the modified initial value problem on $t > 0$

$$y'''' + y = 4\cos(\omega t)\,, \quad \text{with} \quad y(0) = A\,, \quad y'(0) = y''(0) = y'''(0) = 0\,. \tag{5}$$

   Is there a value of A for which $y(t)$ is bounded as $t \to \infty$? Explain your answer clearly.

3. (15 points) Consider the diffusion problem for $u(r, \theta, t)$ in a 2-D disk of radius a with an inflow/outflow flux boundary condition modeled by

$$\begin{gathered} u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\,, \quad 0 \leq r \leq a\,, \quad 0 \leq \theta \leq 2\pi\,, \quad t \geq 0\,, \\ u_r(a, \theta, t) = f(\theta)\,, \quad u \text{ bounded as } r \to 0\,, \quad u \text{ and } u_\theta \text{ are } 2\pi \text{ periodic in } \theta\,, \\ u(r, \theta, 0) = g(r, \theta)\,. \end{gathered}$$

   (a) (3 points) Write the problem that the **steady-state solution** $U(r, \theta)$ would satisfy. Prove that such a steady-state solution $U(r, \theta)$ does not exist when $\int_0^{2\pi} f(\theta)\, d\theta \neq 0$.

   (b) (8 points) Assume that $\int_0^{2\pi} f(\theta)\, d\theta = 0$. Calculate an integral representation for the **steady state solution** $U(r, \theta)$ by summing an appropriate eigenfunction expansion.

   (c) (4 points) Assume that $\int_0^{2\pi} f(\theta)\, d\theta \neq 0$. Calculate an expression for the spatial average of u over the disk, i.e. for $(\pi a^2)^{-1} \int_0^{2\pi} \int_0^a u\, r\, dr d\theta$, and interpret the effect on this average of the net boundary flux $\int_0^{2\pi} f(\theta)\, d\theta$.

Linear Algebra

4. (15 points) Consider the following statements. Either prove the statements are true for all matrices with real entries or provide a counter-example. Note that an orthogonal matrix is square with nonzero, mutually orthogonal columns. A^T denotes the transpose of A.
   (a) (3 points) The product of two $n \times n$ orthogonal matrices is invertible.
   (b) (3 points) The difference between two distinct $n \times n$ orthogonal matrices cannot be singular.
   (c) (3 points) The product of a symmetric matrix and a diagonal matrix is always symmetric.
   (d) (3 points) The Range of an $n \times n$ matrix is perpendicular to its Nullspace.
   (e) (3 points) If A is an $n \times n$ matrix with n odd and $A = -A^T$ then A must be singular.

5. (15 points) Consider real matrices with the block form

$$C = \begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix}$$

   where A is a symmetric square matrix, B^T denotes the transpose of B and B is not necessarily square. The bottom right block is a square matrix of zeros.
   (a) (5 points) Show that C is singular if the number of columns of B is strictly larger than the number of rows.
   (b) (10 points) Show that if A is strictly positive definite, then C is nonsingular iff the columns of B are linearly independent.

6. (15 points) Let $I \in \mathbb{R}^{N,N}$ be the $N \times N$ dimensional identity matrix, where $N \geq 2$ is an integer, and let $\boldsymbol{u} \in \mathbb{R}^N$ and $\boldsymbol{v} \in \mathbb{R}^N$ be any two distinct vectors each with Euclidean length one. Define the matrix A by

$$A = I - \boldsymbol{u}\boldsymbol{v}^T \, .$$

   (a) (5 points) Calculate all the eigenvalues and eigenvectors of A
   (b) (3 points) Prove that A is nonsingular and calculate $\det(A)$.
   (c) (4 points) Derive an explicit formula for A^{-1}.
   (d) (3 points) Let $I \in \mathbb{R}^{N,N}$ for $N \geq 2$ be the identity matrix and define $\boldsymbol{e} \in \mathbb{R}^N \equiv (1, \ldots, 1)^T$ and $\boldsymbol{e}_1 \in \mathbb{R}^N \equiv (1, 0, 0, \ldots, 0)^T$. Prove that the following linear system

$$\left(I - \frac{1}{N} \boldsymbol{e}\boldsymbol{e}^T \right) \boldsymbol{x} = \boldsymbol{e}_1 \, ,$$

   has no solution. Next, if $\boldsymbol{e}_1$ is replaced by an arbitrary vector $\boldsymbol{b}$, what is the condition on $\boldsymbol{b}$ for this problem to have a solution?