The University of British Columbia
Department of Mathematics
Qualifying Examination—Algebra
January 2021

Linear Algebra

1. (15 points) Consider the following statements. Either prove the statements are true for all matrices with real entries or provide a counter-example. Note that an orthogonal matrix is square with nonzero, mutually orthogonal columns. A^T denotes the transpose of A.
   (a) (3 points) The product of two $n \times n$ orthogonal matrices is invertible.
   (b) (3 points) The difference between two distinct $n \times n$ orthogonal matrices cannot be singular.
   (c) (3 points) The product of a symmetric matrix and a diagonal matrix is always symmetric.
   (d) (3 points) The Range of an $n \times n$ matrix is perpendicular to its Nullspace.
   (e) (3 points) If A is an $n \times n$ matrix with n odd and $A = -A^T$ then A must be singular.

2. (15 points) Consider real matrices with the block form

$$C = \begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix}$$

   where A is a symmetric square matrix, B^T denotes the transpose of B and B is not necessarily square. The bottom right block is a square matrix of zeros.
   (a) (5 points) Show that C is singular if the number of columns of B is strictly larger than the number of rows.
   (b) (10 points) Show that if A is strictly positive definite, then C is nonsingular iff the columns of B are linearly independent.

3. (15 points) Let $I \in \mathbb{R}^{N,N}$ be the $N \times N$ dimensional identity matrix, where $N \geq 2$ is an integer, and let $\boldsymbol{u} \in \mathbb{R}^N$ and $\boldsymbol{v} \in \mathbb{R}^N$ be any two distinct vectors each with Euclidean length one. Define the matrix A by

$$A = I - \boldsymbol{u}\boldsymbol{v}^T \,.$$

   (a) (5 points) Calculate all the eigenvalues and eigenvectors of A
   (b) (3 points) Prove that A is nonsingular and calculate $\det(A)$.
   (c) (4 points) Derive an explicit formula for A^{-1}.
   (d) (3 points) Let $I \in \mathbb{R}^{N,N}$ for $N \geq 2$ be the identity matrix and define $\boldsymbol{e} \in \mathbb{R}^N \equiv (1, \ldots, 1)^T$ and $\boldsymbol{e}_1 \in \mathbb{R}^N \equiv (1, 0, 0, \ldots, 0)^T$. Prove that the following linear system

$$\left(I - \frac{1}{N}\boldsymbol{e}\boldsymbol{e}^T \right) \boldsymbol{x} = \boldsymbol{e}_1 \,,$$

   has no solution. Next, if $\boldsymbol{e}_1$ is replaced by an arbitrary vector $\boldsymbol{b}$, what is the condition on $\boldsymbol{b}$ for this problem to have a solution?

Abstract Algebra

4. (15 points) In parts (a) and (b), let $G_x = \{yxy^{-1} \colon y \in G\}$ denote the conjugacy class of x in G.
   (a) (5 points) Let G be a finite group. Show that the cardinality of G_x divides the order of G.
   (b) (5 points) If the group G has order p^r where p is a prime, show that there exists some $x \in G$ other than $x = e$ such that $G_x = \{x\}$. What does this imply about the center of G?
   (c) (5 points) Let G be a group of order ab, where a and b are relatively prime positive integers. Suppose H is a normal subgroup of G of order a. Show that H contains every subgroup of G whose order divides a.

5. (15 points) For all parts of this problem, R denotes the ring $R = \mathbb{Z}[\sqrt{-3}] = \{m + n\sqrt{-3} \colon m, n \in \mathbb{Z}\}$.
   (a) (5 points) Show that $\langle 2 \rangle$, the ideal in R generated by 2, is not a prime ideal.
   (b) (5 points) An element $x \in R$ is called *irreducible* if, whenever $uv = r$ in R, then either u or v is a unit in R. Show that 2 is an irreducible element of R.
   (c) (5 points) Find, with proof, an ideal of R that is not principal.

6. (15 points)
   (a) (4 points) For any rational number q, show that $z = \sin(\pi q)$ is an algebraic number (that is, algebraic over $\mathbb{Q}$).
   (b) (3 points) Suppose E is an algebraic field extension of a field F. Suppose R is a subring of E which contains F. Show that R is a field.
   (c) (4 points) Let $\mathbb{Q}(x)$ be the field of rational functions over $\mathbb{Q}$, and define the two subfields $E = \{f(x) \in \mathbb{Q}(x) \colon f(x) = f(-x)\}$ and $F = \{f(x) \in \mathbb{Q}(x) \colon f(x) = f(2 - x)\}$. (You may assume that these sets are subfields without proving it.) Show that $\mathbb{Q}(x)/E$ and $\mathbb{Q}(x)/F$ are both algebraic extensions and determine their degrees.
   (d) (4 points) In the notation of part (c), is $\mathbb{Q}(x)/(E \cap F)$ an algebraic extension? Justify your answer.