The University of British Columbia
Department of Mathematics
Qualifying Examination—Differential Equations
September 2020

Differential equations

1. (15 points) Consider the following eigenvalue problem for $w(x)$ with eigenvalue parameter λ:

$$\begin{aligned} w''+\gamma w' &= \lambda e^x w\,, \qquad 0 < x < 1\,, \\ w'(0) + \gamma w(0) &= 0\,, \quad w(1) = 0\,. \end{aligned} \tag{1}$$

   Here γ is a real-valued constant satisfying $\gamma > 0$.

   (a) (4 points) Prove that any eigenvalue λ for (1) must be real-valued.

   (b) (4 points) Then, prove that any eigenvalue λ for (1) must satisfy $\lambda < 0$.

   (c) (4 points) State and derive the orthogonality relation for eigenfunctions of (1).

   (d) (3 points) Finally, consider (1) but where the boundary condition on $x = 0$ is now modified to $w'(0) + \gamma w(0) = \lambda w(0)$, where λ is the eigenvalue parameter. By adapting the argument in (a), prove that any eigenvalue to this modified eigenvalue problem is real-valued.

2. (15 points) Consider the initial-value problem, defined on $t \geq 0$, for $y(t)$

$$y''' + 2y'' + y' + y = \sin t\,,$$

   with initial values $y(0) = y'(0) = y''(0) = 0$.

   (a) (4 points) Define $Y(s) = \mathcal{L}(y(t))$, where $\mathcal{L}(y(t))$ denotes the Laplace transform of $y(t)$. Calculate $Y(s)$ explicitly.

   (b) (7 points) By examining the roots of some polynomial related to $Y(s)$ in the half-plane $\text{Re}(s) \geq 0$, prove that $y(t)$ is bounded as $t \to \infty$.

   (c) (4 points) Determine constants a and b such that $y(t) \sim a \sin t + b \cos t$ as $t \to \infty$.

3. (15 points) Consider the 1-D wave equation for $u(x,t)$ satisfying

$$u_{tt} = c^2 u_{xx}\,, \quad -\infty < x < \infty\,, \quad t > 0\,; \qquad u(x,0) = f(x)\,, \quad u_t(x,0) = g(x)\,.$$

   Here $c > 0$ is the constant wave speed.

   (a) (7 points) State and derive D'Alembert's representation formula for $u(x,t)$ in terms of the initial data $f(x)$ and $g(x)$

   (b) (4 points) For $c = 2$, determine $u(x,t)$ explicitly for the initial data

$$f(x) = 0\,, \quad -\infty < x < \infty\,; \qquad g(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}\,.$$

   (c) (4 points) For your solution in (b), give a careful plot of $u(x,t)$ versus x for $t = 1$ and for $t = 4$.

Linear Algebra

4. (15 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3 & 1 & 3 \end{bmatrix}.$$

   (a) (2 points) Calculate the trace of A.

   (b) (2 points) Calculate the determinant of A.

   (c) (4 points) What is the nullity of A (the dimension of the null space)?

   (d) (2 points) What is the rank of A?

   (e) (5 points) Write a basis for the nullspace of A.

5. (15 points) Consider the problem of finding polynomials $B_n(x)$ with real coefficients such that

$$\int_x^{x+1} B_n(t)\, dt = x^n.$$

   (a) (4 points) Find a polynomial B_1 with this property.

   (b) (4 points) Find a polynomial B_2 with this property.

   (c) (7 points) Show that there is a unique polynomial $B_n(x)$ with this property for all n.

6. (15 points) Let V be a finite dimensional vector space over the real numbers. Let $(\mathbf{x}, \mathbf{y})$ be an inner product for V and let L be a linear functional on V $(L : V \to \mathbb{R})$.

   (a) (5 points) Write the properties that define a linear functional in this setting.

   (b) (10 points) Show that there exists a unique vector $\mathbf{y}$ in V such that

$$L(\mathbf{x}) = (\mathbf{x}, \mathbf{y})$$

   for all $\mathbf{x}$.