The University of British Columbia
Department of Mathematics
Qualifying Examination—Analysis
September 2020

In the real and complex analysis parts of this exam, please state carefully any results that you use in your arguments

Real analysis

1. (a) (2 points) Let K be a subset of a metric space (M, d). Suppose that for every $\epsilon > 0$, one can cover K by finitely many ϵ-balls in M. Does it follow that the closure $\overline{K}$ of K is compact? Either prove or give a counterexample.

   (b) (4+4=8 points) Let

$$\ell^2(\mathbb{C}) := \Big\{ x = (x_1, x_2, \ldots) : x_j \in \mathbb{C},\ ||x||_2^2 := \sum_{j=1}^{\infty} |x_j|^2 < \infty \Big\}$$

   denote the metric space of infinite square summable sequences, with the distance function $d(x, y) = ||x - y||_2$. For $\alpha = 1$ and $\alpha = 2$, determine whether the set

$$K_\alpha = \Big\{ x \in \ell^2(\mathbb{C}) : |x_n| \le n^{-\frac{\alpha}{2}} \text{ for all } n = 1, 2, \ldots \Big\}$$

   is compact in $\ell^2(\mathbb{C})$.

2. Determine whether the following statements are true or false, with adequate justification.

   (a) (2 points) Let $\mathbb{T}$ denote the unit circle on the plane centred at the origin. If $f : \mathbb{T} \to \mathbb{C}$ is a continuous function for which

$$\oint_{\mathbb{T}} z^n f(z)\, dz = 0 \text{ for all } n = 0, 1, 2, \cdots,$$

   then $f \equiv 0$ on $\mathbb{T}$.

   (b) (3 points) Let $\mathcal{C}^1[a, b]$ denote the space of continuously differentiable real-valued functions on $[a, b]$, equipped with the norm

$$||f||_{\mathcal{C}^1} := \sup_{x \in [a,b]} |f(x)| + \sup_{x \in [a,b]} |f'(x)|.$$

   Then every bounded subset of $\mathcal{C}^1([a, b])$ admits a uniformly convergent subsequence.

   (c) (5 points) Let $\mathbb{N}$ denote the set of positive integers. There exists an uncountable collection $\{\mathbb{N}_i : i \in \mathbb{I}\}$ of distinct infinite subsets of $\mathbb{N}$ such that $\mathbb{N}_i \cap \mathbb{N}_j$ is finite for all $i, j \in \mathbb{I}$, $i \ne j$.

3. (a) (3 points) Specify a class of functions $f : \mathbb{R}^2 \to \mathbb{R}$ (which is strictly larger than the class of all bivariate polynomials) and a family of curves Γ for which

$$\oint_{\Gamma} (f_y dx + f_x dy) = 0.$$

   (b) (3.5 + 3.5 = 7 points) Evaluate

$$\oint_{\Gamma} \frac{x dy - y dx}{x^2 + y^2},$$

   for two choices of Γ:

$$\Gamma = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \quad \text{and} \quad \Gamma = \{(x, y) \in \mathbb{R}^2 : (x - 2)^2 + y^2 = 1\}.$$

   In both cases, assume that Γ is oriented counterclockwise.

Complex analysis

4. Let $D_R = \{z \in \mathbb{C} : |z| < R\}$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in D_R. Let $u(z) = \mathrm{Re}(f(z))$.

   (a) (5 points) Prove that for all $n \in \mathbb{N}$ and $0 < r < R$,

   $$a_n = \frac{1}{\pi r^n} \int_0^{2\pi} u(r\mathrm{e}^{\mathrm{i}t}) \mathrm{e}^{-\mathrm{i}nt} dt.$$

   (b) (5 points) Assume that $f(0) \in \mathbb{R}$. Prove that for any $0 < r < R$ and $|z| < r$,

   $$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(r\mathrm{e}^{\mathrm{i}t}) \frac{r + z\mathrm{e}^{-\mathrm{i}t}}{r - z\mathrm{e}^{-\mathrm{i}t}} dt\,.$$

   Hint. $(r - z\mathrm{e}^{-\mathrm{i}t})^{-1}$ is analytic.

5. Let $\Omega \subset \mathbb{C}$ be open and such that $D_R = \{z \in \mathbb{C} : |z| < R\} \subset \Omega$. Let f be holomorphic in Ω and assume that

   $$M = \sup\{|f(z)| : |z| \leq R\} > 0.$$

   (a) (5 points) Let $|z| < \frac{R}{M}$. Show that the equation

   $$\zeta = z f(\zeta)$$

   has a unique solution in D_R. Denote this solution by $\zeta = g(z)$.

   (b) (5 points) Let γ be the positively oriented circle of radius R, centred at the origin. Prove that

   $$g(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{\zeta(1 - z f'(\zeta))}{\zeta - z f(\zeta)} d\zeta\,.$$

6. (10 points) Let $m, n \in \mathbb{N}$ be such that $0 < m < n$. Prove that

   $$\int_0^\infty \frac{x^{m-1}}{1 + x^n} dx = \frac{\pi/n}{\sin(\pi m/n)}.$$

   Hint. Consider the boundary of the sector $\{r\mathrm{e}^{\mathrm{i}\theta} : 0 \leq \theta \leq 2\pi/n, 0 \leq r \leq R\}$.