The University of British Columbia
Department of Mathematics
Qualifying Examination—Algebra
September 2020

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1. (15 points) Consider the matrix
$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3 & 1 & 3 \end{bmatrix}.$$
   (a) (2 points) Calculate the trace of A.
   (b) (2 points) Calculate the determinant of A.
   (c) (4 points) What is the nullity of A (the dimension of the null space)?
   (d) (2 points) What is the rank of A?
   (e) (5 points) Write a basis for the nullspace of A.

2. (15 points) Consider the problem of finding polynomials $B_n(x)$ with real coefficients such that
$$\int_x^{x+1} B_n(t)\, dt = x^n.$$
   (a) (4 points) Find a polynomial B_1 with this property.
   (b) (4 points) Find a polynomial B_2 with this property.
   (c) (7 points) Show that there is a unique polynomial $B_n(x)$ with this property for all n.

3. (15 points) Let V be a finite dimensional vector space over the real numbers. Let $(\mathbf{x}, \mathbf{y})$ be an inner product for V and let L be a linear functional on V ($L : V \to \mathbb{R}$).
   (a) (5 points) Write the properties that define a linear functional in this setting.
   (b) (10 points) Show that there exists a unique vector $\mathbf{y}$ in V such that
$$L(\mathbf{x}) = (\mathbf{x}, \mathbf{y})$$
   for all $\mathbf{x}$.

4. (15 points) In parts (a) and (b) of this question, $Z(G)$ denotes the center of the group G, that is, the set of elements that commute with every element of G.
   (a) (4 points) Let G be a group such that $G/Z(G)$ is cyclic. Prove that there exists $x \in G$ such that every element of G can be written as $x^n z$ for some $n \in \mathbb{Z}$ and some $z \in Z(G)$.
   (b) (3 points) If G is a group such that $G/Z(G)$ is cyclic, prove that G is abelian.
   (c) (4 points) Let G be a finite group, and let p be a prime that divides the order of G. Let H be a subgroup of G of index p. Define $K = \{g \in G \colon (gx)H = xH \text{ for all } x \in G\}$. Prove that K is a normal subgroup of G, and prove that the order of G/K divides $p!$. (Hint: there is a relevant group action of G on the set of cosets $\{xH \colon x \in G\}$ given by $xH \mapsto (gx)H$.)
   (d) (4 points) Let G be a finite group, and let p be the *smallest* prime dividing the order of G. If H is a subgroup of G of index p, prove that H is a normal subgroup of G.

5. (15 points) In this question, R is a commutative ring with 1. Recall that an element a of R is *nilpotent* if there exists a positive integer n such that $a^n = 0$.

   (a) (5 points) Let J be the set of nilpotent elements of R. Prove that J is an ideal of R that is contained in every prime ideal of R.

   (b) (5 points) Given $y, z \in R$, prove that $y + zT$ is a unit in $R[T]$ if and only if y is a unit in R and z is nilpotent.

   (c) (5 points) Suppose that R is finite. Prove that every nonzero element of R is either a unit or a zero divisor.

6. (15 points) For parts (a) and (b) of this question, let p be a prime, let $\mathbb{F}_p$ be the field with p elements, and fix $a \in \mathbb{F}_p \setminus \{0\}$.

   (a) (3 points) Consider the polynomial $f(T) = T^p - T + a \in \mathbb{F}_p[T]$. Prove that if α is a root of $f(T)$ in some extension of $\mathbb{F}_p$, then so is $\alpha + 1$.

   (b) (4 points) What is the Galois group of the splitting field of the polynomial $f(T^p) = T^{p^2} - T^p + a$ over $\mathbb{F}_p$?

   (c) (4 points) Find, with proof, the Galois group of the extension $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$.

   (d) (4 points) Prove that the set $\{\sqrt{2}, \sqrt{3}, \sqrt{6}\}$ is linearly independent over $\mathbb{Q}$.