The University of British Columbia
Department of Mathematics
Qualifying Examination—Analysis
January 5, 2019

Real analysis

1. (10 points) Suppose that $\{f_n\}$ is a sequence of continuous functions on $[0,1]$ that converges pointwise to a continuous function f on $[0,1]$. Does it follow that $\{f_n\}$ converges uniformly? If so, give a proof. If not, provide a counter-example.

2. (10 points) Let f and g be functions that are absolutely continuous on $[0,1]$ and positive for each $x \in [0,1]$. Prove that the quotient f/g is absolutely continuous on $[0,1]$.

3. (10 points) Let $f\colon [0,1]^n \to \mathbb{R}$ be a smooth function that is supported on the unit cube in $\mathbb{R}^n$ and vanishes on the boundary of the cube. Prove that

$$\int_{[0,1]^n} |f(x)|^2 dx \leq \int_{[0,1]^n} |\nabla f(x)|^2 dx.$$

Note: this is an example of a Poincaré inequality, but it is not sufficient to merely cite this fact; you need to prove it.

Complex analysis

4. Let $f(z) = \left(\frac{\sin(2z)}{z^3} - \frac{2}{z^2}\right) \cdot \left(\frac{z+\pi/4}{z-\pi/4}\right)$.
   (a) Find and classify all singularities of f.
   (b) Evaluate $I = \int_\Gamma f(z)dz$ where Γ is the positively oriented rectangular loop with vertices at $v_1 = i$, $v_2 = -1$, and $v_3 = -i$, and $v_4 = 1$.

5. Let $f : \mathbb{C} \mapsto \mathbb{C}$ be analytic inside and on a circle C_R of radius $R > 0$ centered about z_0.
   (a) Show that if $|f(z)| \leq M$ for all z on C_R, then

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{R^n}.$$

   (b) Use part (a) to prove Liouville's theorem: "The only bounded entire functions are the constant functions."
   (c) Let f be entire and suppose that $f^{(4)}$ is bounded in the whole complex plane. Prove that f must be a polynomial of degree at most 4.

6. Evaluate the following integrals using contour integration.
   (a)

$$I = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx$$

   (b)

$$J = \int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}$$

   (c)

$$K = \int_0^{\infty} \frac{\cos(x)}{x^2+4} dx$$