The University of British Columbia
Department of Mathematics
Qualifying Examination—Algebra
January 11, 2020

This exam consists of problems 1 to 3 on Linear Algebra and problems 4 to 6 on Abstract Algebra. The two subjects will be weighted equally.

---

1. (15 points) Consider the matrix
$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 3 & 4 \end{bmatrix}.$$
   (a) (3 points) What is the rank of A?
   (b) (4 points) Find a basis for the nullspace of A^T (the transpose of A).
   (c) (6 points) Determine all values of w for which the system
$$x + y = -1 \quad (1)$$
$$x + 2y = w \quad (2)$$
$$3x + 4y = 0 \quad (3)$$
   has a solution and find *one*.
   (d) (2 points) Is the solution in (c) above unique?

2. (15 points) Consider the sequence $\{\mathbf{v}^n\}_{n=0}^{\infty}$ of vectors in $\mathbb{R}^2$ defined by given $\mathbf{v}^0$ and the recurrence relationship
$$A\mathbf{v}^{n+1} = B\mathbf{v}^n$$
   where
$$A = \begin{bmatrix} 1 & -k \\ k & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & k \\ -k & 1 \end{bmatrix}$$
   and $k > 0$ is a given parameter.
   (a) (8 points) Rewite the recurrence relationship in the form
$$\mathbf{v}^{n+1} = C\mathbf{v}^n$$
   with C a matrix.
   (b) (7 points) Show that $\|\mathbf{v}^n\| = \|\mathbf{v}^0\|$ for all n where $\|\cdot\|$ is the standard Euclidean norm.

3. (15 points) (a) (7 points) Prove that
$$e^A := I + A + A^2/2! + \cdots + A^m/m! + \dots$$
   converges at every index for any square matrix. Here, I is the identity matrix.
   (b) (4 points) Find e^A when
$$A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$
   (c) (4 points) Find e^A when
$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

For problems 4 to 6, complete parts (a)–(c) of *each problem*, and complete part (d) of *one problem of your choice*. (For example, if you choose to complete part (d) of problem 4, then problem 4 will be worth 19 points and problems 5 and 6 will be worth 13 points each, for a total of 45 points.) You are free to attempt one part (d) below or more than one; the part (d) for which you receive the highest score will be the one that is counted.

4. (13 or * 19 points) In this question, the rings R and S are always commutative and have a multiplicative identity 1 satisfying $1 \neq 0$. Prove or disprove each of the following statements involving direct products of rings:
   (a) (4 points) If R^* denotes the set of invertible elements in R, then $(R \times S)^* = R^* \times S^*$.
   (b) (4 points) If R and S are fields, then $R \times S$ is a field.
   (c) (5 points) If K is an ideal of $R \times S$, then there exist an ideal I of R and an ideal J of S such that $K = I \times J$.
   (d) ($*$ 6 points) If $e \in R$ is an element satisfying $e^2 = e$, then $R \cong R/\langle e\rangle \times R/\langle 1-e\rangle$.

5. (13 or * 19 points) It is known that any nontrivial finite abelian group G has an *invariant factor decomposition* as a direct sum of finite cyclic groups, of the form

$$G \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_\ell\mathbb{Z}, \qquad (\dagger)$$

   where $\ell \geq 1$ and $d_1, \ldots, d_\ell \geq 2$ are integers such that d_j divides d_{j+1} for each $1 \leq j \leq \ell - 1$.
   (a) (4 points) Let $\mathbb{F}_q$ denote the finite field with q elements, and let R^* denote the set of invertible elements in the ring R. Then $H = (\mathbb{F}_{64} \times \mathbb{F}_{27} \times \mathbb{F}_{25} \times \mathbb{F}_{49})^*$ is a finite abelian group. Find the invariant factor decomposition of H.
   (b) (4 points) Recall that the *exponent* of an additive group G is the smallest positive integer n such that $ng = 0$ for each element g of G. In the notation ($\dagger$), prove that the largest invariant factor d_ℓ is equal to the exponent of the group G.
   (c) (5 points) Prove that the invariant factor decomposition of a finite abelian group is unique. (You don't have to prove that it exists.)
   (d) ($*$ 6 points) Let $m_1, \ldots, m_k$ be any positive integers, and set $G = \mathbb{Z}/m_1\mathbb{Z} \oplus \mathbb{Z}/m_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_k\mathbb{Z}$. Then G has an invariant factor decomposition ($\dagger$) of length ℓ. Prove that $\ell \leq k$.

6. (13 or * 19 points) Define the fields $K = \mathbb{Q}(\sqrt[3]{2})$ and $L = \mathbb{Q}(e^{2\pi i/3})$, and let F be the splitting field over $\mathbb{Q}$ of the polynomial $(x^3 - 2)(x^3 - 3)$.
   (a) (7 points) Among the four fields $F, K, L, \mathbb{Q}$, determine (with justification) all pairs of fields where one is a field extension of another.
   (b) (3 points) For the smallest field extension from part (a), either determine its Galois group or explain why it is not a Galois extension.
   (c) (3 points) Do the same for the second-smallest field extension from part (a).
   (d) ($*$ 6 points) Do the same for the remainder of the field extensions from part (a).