The University of British Columbia
Department of Mathematics
Qualifying Examination—Differential Equations
September 2019

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1. (15 points) Consider the following system of nonlinear ODEs for $x(t)$ and $y(t)$ in the region $-\infty < x < \infty, -\infty < y < \infty$:
$$x' = x\,[4x(1-x) - y]\ , \qquad y' = y(4x-3)\,.$$
   (a) Find all fixed points, determine their linear stability properties, and classify the fixed points.

   (b) Use the eigen-analysis of the linearized system in (a) to help sketch a local phase portrait near each fixed point.

   (c) Sketch the global phase portrait. Explain any discrepancy between the portraits of the linearized and nonlinear systems. (Hint: the nullclines will be useful here).

2. (15 points) Consider the differential operator L, defined as follows for functions $y(x)$ with domain $x > 0$:
$$L[y] = x^2 y'' - xy' - 3y\,.$$
   (a) Find the general solution for $L[y] = 0$.

   (b) Express the ODE $L[y] = 0$ in matrix form as
$$\frac{d\vec{y}}{dx} = A(x)\vec{y}\,.$$
   Find a fundamental matrix Y for this problem.

   (c) Use the results from part (b) to solve the initial value problem
$$L[y] = 0\,, \qquad \text{with} \qquad y(1) = 0\,, \quad y'(1) = -1\,.$$
   (d) Find the general solution for the inhomogeneous problem $L[y] = f$ for the choice $f = x^3$.

3. (15 points) Radially symmetric diffusion in an insulated sphere of radius a, and with constant bulk decay, is modeled by the following parabolic PDE for $u = u(r,t)$:
$$u_t = D\left[u_{rr} + \frac{2}{r}u_r\right] - u\,, \quad 0 < r < a\,,\ t > 0\,, \tag{1a}$$
$$u_r(a,t) = 0\,; \qquad u\,,\, u_r \text{ bounded as } r \to 0\,, \tag{1b}$$
$$u(r,0) = f(r)\,. \tag{1c}$$
   Here $D > 0$ is the constant diffusivity.

   (a) Define $M(t)$ by $M(t) = \int_0^a r^2 u(r,t)\,dr$. Derive an ODE for $M(t)$ and solve it to determine $M(t)$ in terms of an integral of $f(r)$.

   (b) Develop, in as explicit a form as you can, an eigenfunction expansion representation for the solution $u(r,t)$ to (1). Verify from your representation the result obtained in (a) above. (Hint: In determining the eigenvalues and eigenfunctions of the underlying Sturm-Liouville eigenvalue problem for $\Phi(r)$, the change of dependent variable $\Phi(r) = \Psi(r)/r$ is particularly useful.)

4. (15 points) (a) Work over the complex numbers. Let $A = \begin{pmatrix} 3 & -2 \\ 8 & -5 \end{pmatrix}$. Find the eigenvalues and state their geometric and algebraic multiplicities.

   (b) Is the matrix above diagonalizable? Explain your answer.

   In either case, write down a similarity transform putting A in Jordan normal form.

   (c) Let B be a real matrix with characteristic polynomial $(x+2)^2(x-3)^2$. What are the possible Jordan normal forms of B? To avoid repetition, give your answers with eigenvalues sorted from smallest-in-magnitude to largest-in-magnitude, and if two Jordan forms J_1 and J_2 happen to be similar matrices, give only one. You may omit 0-entries if you wish.

5. (15 points) (a) Let P_n be the vector space of polynomials of degree at most n with real coefficients. Let $S = \{p_1, \ldots, p_{n+1}\} \subseteq P_n$ be a set of $n+1$ polynomials, satisfying $p_i(0) = 0$ for all i. Either prove S is linearly dependent, or give an example to show S may be linearly independent.

   (b) Let $\vec{u}$ and $\vec{v}$ be elements of an inner product space. Suppose that

$$|\vec{u} + \vec{v}| = |\vec{u}| + |\vec{v}|.$$

   Show that $\vec{u}$ and $\vec{v}$ are linearly dependent. Name, or otherwise state clearly, any theorems that you use.

6. (15 points) Consider a real 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

   (a) Let m be a root (real or complex) of the "*sector equation*" $bm^2 + (a-d)m - c = 0$. Show that $\vec{v} = \begin{pmatrix} 1 \\ m \end{pmatrix}$ is an eigenvector of A and determine the corresponding eigenvalue.

   (b) What practical advantage might this method offer over the "usual" characteristic-equation approach to finding eigenvalues and eigenvectors?

   (c) Consider the real matrix $B(\varepsilon) = \begin{pmatrix} 1 & \varepsilon \\ 1 & 2 \end{pmatrix}$, where ε is a small parameter.

      i. Find the roots of the sector equation from above.

      ii. Describe carefully what happens to each of the roots as $\varepsilon \to 0$. (Hint: the Taylor expansion $\sqrt{1+\tau} = 1 + \frac{1}{2}\tau + O(\tau^2)$ for small τ may be useful here.)

      iii. Nonetheless, show that all roots lead to finite eigenpairs $(\lambda, \vec{v})$ in the limit as $\varepsilon \to 0$. Hint: recall any scaling of the eigenvector is also an eigenvector.