The University of British Columbia
Department of Mathematics
Qualifying Examination—Analysis
January 5, 2019

Real analysis

1. (10 points) 1. (a) Suppose $f(x, y)$ is a continuously differentiable function on $\mathbb{R}^2$. It is known that the directional derivatives satisfy $D_{\mathbf{u}} f(0,0) < \frac{\partial f}{\partial x}(0,0) = 2$ for all unit vectors $\mathbf{u} \neq \mathbf{i}$. Find $\frac{\partial f}{\partial y}(0,0)$.

   (b) Let $\mathcal{C}$ be the boundary of the parallelogram in the $x - y$ plane with vertices $(0,0)$, $(2,0)$, $(3, y_0)$ and $(1, y_0)$, where $y_0 > 0$ is unknown. $\mathcal{C}$ is given the counterclockwise orientation. Suppose $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 6$, where $\mathbf{F} = (-y + e^{x^2} + e^{y^2}, 2x + 2xye^{y^2})$. Find y_0.

2. (10 points) Are the following true or false? If true, give a proof; if false, provide a counterexample.
   (a) If $f, g : (0,1) \to \mathbb{R}$ are uniformly continuous on $(0,1)$, then so is $h(x) = f(x)g(x)$.
   (b) If $f, g : (0,1) \to (0,\infty)$ are uniformly continuous and positive on $(0,1)$, then so is $h(x) = f(x)/g(x)$.
   (c) If f is a continuously differentiable function on $[0,1]$, then there is a sequence of polynomials $\{P_n : n \in \mathbb{N}\}$ such P_n converges uniformly to f, and P_n' converges uniformly to f'.

3. (10 points) Let $K(x, y) = \sin(2\pi(x - y))^2$.
   (a) If $f : [0,1] \to \mathbb{R}$ is continuous, prove that $F(x) = \int_0^1 K(x,y) f(y)\, dy$ defines a continuous function F on $[0,1]$.
   (b) Prove that there is a unique continuous function $f : [0,1] \to \mathbb{R}$ such that

$$f(x) = x + \int_0^1 K(x,y) f(y)\, dy \quad \text{for all } x \in [0,1].$$

Complex analysis

4. (10 points) Let

$$f(z) = (z^2 + 1)^2, \qquad g(z) = (z^2 + 2z - 3)^3, \qquad h(z) = \frac{f(z)}{g(z)},$$

   and let $\mathcal{C}$ be the circle $|z| = 2$ with the counter-clockwise orientation. Find $\oint_{\mathcal{C}} \frac{h'z)}{h(z)} + f(z)g(z)dz$.

5. (10 points) Use complex integration to compute the following integrals:

$$\text{(a)} \quad I_1 = \int_{-\infty}^{\infty} \frac{1}{1 + x^4} dx \qquad \text{(b)} \quad I_2 = \int_{-\infty}^{\infty} \frac{\cos(x)}{1 + x^4} dx$$

6. (10 points) (a) Find a harmonic function g on the upper half-plane $U = \{z : Im(z) > 0\}$ which extends continuously to $\bar{U} \setminus \{0\}$ and satisfies the boundary conditions $g = 0$ on $(0, \infty)$ and $g = 1$ on $(-\infty, 0)$.
   (b) If D is the half disk $\{|z| < 1 : Im(z) > 0\}$, find a conformal map f from $\bar{D}$ to $\bar{U} \cup \{\infty\}$ such that $f(1) = \infty$ and $f(-1) = 0$.
   (c) Find a solution, h, to $\Delta h = 0$ on D which extends continuously to $\bar{D} \setminus \{-1, 1\}$ and satisfies $h = 0$ on $(-1, 1)$ and $h = 1$ on the semicircle $\{|z| = 1, Im(z) > 0\}$. (You may express your answer in terms of the solutions of previous parts.)