The University of British Columbia
Department of Mathematics
Qualifying Examination—Algebra
January 5, 2019

This exam consists of problems 1 to 3 on Linear Algebra and problems 4 to 6 on Abstract Algebra. The two subjects will be weighted equally.

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1. (12 points) Some questions on eigenvalues.
   (a) For the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, determine the eigenvalues, their algebraic multiplicities, and their geometric multiplicities.
   (b) A 2×2 matrix B has eigenvalues $1+i$ and $1-i$. Find B^4 and explain your answer.

2. (15 points) Consider a 2×2 matrix A with real entries.
   (a) What are the possible values for $\text{rank}(A)$?
   (b) For each of the values of $\text{rank}(A)$ in (a), make two schematic diagrams (sketches): the first should show the nullspace (or kernel) of A; the second should show the column space (or range) of A.
   (c) For a full-rank A, make a schematic diagram (sketch) showing the image of the unit circle $x^2+y^2=1$ under the transform $A\begin{bmatrix} x \\ y \end{bmatrix}$.
   (d) Give the definition of the matrix 2-norm $\|A\|_2$. Draw a diagram illustrating the definition.
   (e) Consider s as the set of all points on a line segment. What is the image of s under transformation by A? Justify your answer.
   (f) Sketch the set of points (x, y) in $\mathbb{R}^2$ where $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ has $\|\vec{v}\|_1 = 1$.
   (g) Sketch the image of your previous answer under transformation by A. Indicate $\|A\|_1$ on your sketch.

3. (12 points) Recall that an $n \times n$ *orthogonal matrix* Q is a matrix where the columns form an orthonormal basis of $\mathbb{R}^n$.
   (a) What is Q^TQ? Justify your answer.
   (b) What is Q^{-1}?
   (c) Suppose we are given a vector b, an orthogonal matrix Q, and an upper triangular matrix R, such that $A = QR$. Describe a simple way to solve $Ax = b$ (assuming such an x exists) without explicitly forming A and without using Gaussian Elimination.
   (d) Continue to assume that $A = QR$ is as in (c), and define $B = RQ$. What can be said about the eigenvalues of A and the eigenvalues of B? Justify your answer.

4. (10 points) Let R be a commutative ring with unit, and denote by F the set of all functions $f : R \to R$— not necessarily homomorphisms. The set F can be endowed with a ring structure by pointwise addition and multiplication

$$(f+g)(r) = f(r) + g(r) \quad (fg)(r) = f(r)g(r).$$

There is a homomorphism $e_R : R[t] \to F$ given by $e_R(p)(r) = p(r)$. That is, $e_R(p)$ is the function defined by the polynomial p.

(a) Find, with proof, a ring S such that e_S is not injective.

(b) For any element $r \in R$, define an *indicator function* $1_r \in F$ by the formula:

$$1_r(x) = \begin{cases} 1 & \text{if } r = x; \\ 0 & \text{otherwise;} \end{cases}$$

Suppose 1_r is in the image of e_R and $r \neq 0$. Show that r is invertible in R.

(c) Suppose the homomorphism e_R is surjective. Prove that R is a finite field.

5. (10 points) (a) Give, without proof, the isomorphism classes of all groups H of order 25. For each H, determine (with proof) the order of the group $\mathrm{Aut}(H)$, the group of automorphisms of the group H.

(b) Suppose G is a non-abelian group of order 75. Determine, with proof, the possible isomorphism classes of the Sylow 5 subgroup of G.

(c) Construct a non-abelian group of order 75.

6. (10 points) Suppose $f(x) \in \mathbb{Q}[x]$ is an irreducible polynomial. Let $E/\mathbb{Q}$ denote the splitting field, and suppose $\alpha, \beta \in E$ are two distinct roots of $f(x)$ such that $\alpha + \beta \in \mathbb{Q}$. Prove that the degree of f is even.