The University of British Columbia
Department of Mathematics
Qualifying Examination—Differential Equations
September, 2018

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1. (7 points) Find all eigenvalues and a basis for each eigenspace of the operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x, y, z) = (2x + y, y - z, 2y + 4z).$$

2. (8 points) Let $\vec{u} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$. Is $A := \vec{u}\,\vec{u}^T \in \mathcal{M}_n(\mathbb{R})$ diagonalizable?

3. (10 points) Let $A = \begin{pmatrix} 2 & -2 & 1 \\ 2 & -3 & 2 \\ 1 & 2 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$.
   a) Is A diagonalizable?
   b) Give a basis for the $\mathbb{R}$-vector space $\mathcal{C}(A) := \{B \in \mathcal{M}_3(\mathbb{R}),\ AB = BA\}$.

4. (14 points) Consider the differential equation (DE)

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0,$$

   with constant coefficients a, b, c. You are told that one solution of this DE is $y_1(t) = e^{-2t}$, but the other solution is to be found. You also know the Wronskian of the solutions is

$$W(y_1, y_2) = 5e^t.$$

   (a) Explain in 1-2 sentences what the Wronskian tells us about the solutions to the DE.
   (b) Find the second solution, $y_2(t)$.
   (c) If $a = 1$, what are the values of the constants b, c?
   (d) Determine the solution to the initial value problem given the initial conditions $y(0) = 1, y'(0) = 8$.
   (e) Does the solution in (d) ever change sign for $t > 0$? If so, at what time; if not, why not?

   Recall that the Wronskian is defined as

$$W(y_1, y_2) \equiv \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.$$

5. (16 points) Consider the following initial value problem (IVP):

$$\frac{\partial u}{\partial t} - 3x\frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty, \quad t > 0,$$

$$\text{with} \quad u(x, 0) = \begin{cases} x(4 - x) & \text{if } 0 \le x \le 4, \\ 0 & \text{if } x < 0 \text{ or } x > 4. \end{cases}$$

   You are asked to solve this IVP using the **Method of Characteristics**.

   (a) Show that along the appropriate "characteristic curves" $x = f(x_0, t)$, the PDE can be rewritten as

$$\frac{du}{dt} = 0.$$

(b) Find the equations of the characteristic curves and sketch them in the x versus t plane.

(c) Solve the IVP, that is, find $u(x,t)$.

(d) Interpret the behaviour of the solution, that is, say verbally what would be seen by an observer measuring u at some location x as time increases.

(e) Calculate the total mass as a function of time t,

$$U(t) = \int_{-\infty}^{\infty} u(x,t)dx.$$

6. (15 points) Consider the following initial boundary value problem on $0 \leq x \leq L$ and $t \geq 0$ for the concentration of a chemical $c(x,t)$ inside a pipe of length L:

$$c_t = Dc_{xx}, \quad c(x,0) = \phi(x), \quad c_x(0,t) = A, \quad c(L,t) = B,$$

where A and B are constants.

(a) Provide a physical interpretation of the above problem, by explaining what is happening inside the pipe and at its two ends.

(b) What will be the chemical profile in the pipe after a long time? Determine the form of the distribution $c_{ss}(x)$ as $t \to \infty$. Are there any conditions on A, B, D, ϕ that need to be satisfied for this solution to exist physically?

(c) Define $u(x,t) = c(x,t) - c_{ss}(x)$. What initial boundary value problem does u satisfy?

(d) Use Separation of Variables to solve the problem in part (c). Leave your answer in terms of the constants in the original problem and the initial condition.