Algebra Qualifying Exam

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1. (8 points) Let p be a prime number, $n \in \mathbb{N}$, with $n \geq 1$ and $M \in \mathcal{M}_n(\mathbb{Z})$. Show that

$$\operatorname{tr}(M^p) \equiv \operatorname{tr}(M) \mod p\ .$$

2. (10 points) Let $A = \begin{pmatrix} 2 & -2 & 1 \\ 2 & -3 & 2 \\ 1 & 2 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$.
   a) Is A diagonalizable?
   b) Give a basis for the $\mathbb{R}$-vector space $\mathcal{C}(A) := \{B \in \mathcal{M}_3(\mathbb{R}),\ AB = BA\}$.

3. (7 points) Let $\vec{u} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$. Is $A := \vec{u}\,\vec{u}^T \in \mathcal{M}_n(\mathbb{R})$ diagonalizable?

4. (15 points) Let k be a field and $P \in k[X]$ with degree $n \geq 2$.
   a) Show that P is irreducible over k if and only if P has no root in the extensions of k of degree $\leq n/2$.
   b) Show that $X^4 + 1$ has a root in $\mathbb{F}_{p^2}$ and that it is reducible over $\mathbb{F}_p$ for any prime number p.

5. (15 points) Let p be a prime number and ε a p^{th} primitive root of 1 in $\mathbb{C}$. We admit that the minimal polynomial of ε over $\mathbb{Q}$ is $\Phi_p = 1 + X + ... + X^{p-1}$.
   a) Compute the Euclidean division of Φ_p by $X - 1$.
   b) Let $A = \mathbb{Z}[\varepsilon]$ be the subring of $\mathbb{C}$ generated by ε. Show that A is a free abelian group of rank $p - 1$.
   c) Show that $\mathbb{Z} \cap (1 - \varepsilon)A = p\mathbb{Z}$.

6. (15 points) Let p be prime number and $G = \mathrm{GL}_2(\mathbb{F}_p)$. How many p-Sylow subgroups does G have?