The University of British Columbia
Department of Mathematics
Qualifying Examination—Algebra
January 6, 2018

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1. (10 points) Find an $n \times n$-matrix P with real entries, such that $P^T = P$, $P^2 = P$, and whose null space is spanned by the vector $(1, \ldots, 1)^T$.

2. (10 points) Let A be an $n \times n$ matrix with complex entries. Suppose that m is a positive integer such that A^m is diagonalizable. Prove that A^{m+1} is diagonalizable.

3. (10 points) Suppose that A is a 2×2 matrix with real entries. Suppose that $\det A = 1$, and that A does not have a real eigenvalue. Prove that there exists an invertible 2×2-matrix S, with real entries, such that $S^{-1}AS$ is equal to the rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, for some $\theta \in (0, \pi) \subset \mathbb{R}$.

4. (10 points) Let G be a finite group, such that for every subgroup H of G, there exists a homomorphism $\phi : G \to H$, such that $\phi(x) = x$, for all $x \in H$. Prove that G is a product of cyclic groups of prime order.

5. (10 points) Let p be a prime number. Prove that there are exactly $\frac{1}{p}(2^p - 2)$ irreducible polynomials of degree p in one variable over the field with 2 elements.

6. (10 points) Determine the structure of the Galois group of the polynomial $x^4 - 4x^2 + 2$ of the rationals.