UBC Department of Mathematics Qualifying Exam in Analysis

January, 2017

Every problem is worth 10 points.

Problem 1: Let D be the open unit disc, ∂D its boundary and $\bar{D}$ its closure. Let f_n be a sequence of functions holomorphic in D, continuous on $\bar{D}$, and converging uniformly on ∂D. Show that f_n converges uniformly on $\bar{D}$ (you may use that a sequence a_n converges uniformly iff $\{a_n\}$ is uniformly Cauchy).

Problem 2: Use residues to calculate $\int_{-\infty}^{\infty} \frac{1}{1+x^3} dx$

Problem 3: Consider the upper half plane $U = \{z : Imz \geq 0\}$. Let f be continuous on the closure $\bar{U}$ such that $f(x)$ is real for x real, and f is holomorphic on U. Let $F(z)$ be the extension of f to lower half plane defined by $F(z) = \overline{f(\overline{z})}$ where $Imz < 0$. Show that $F(z)$ is entire. (you may assume $F(z)$ is continuous. You may also assume that in the statement of Moreras theorem, the closed loops are all rectangles).

Problem 4:

For each of the following vector fields $\mathbf{F}$, determine if $\mathbf{F}$ is conservative on $\mathbb{R}^3$. For each that is conservative (i.e., a gradient vector field), find all potentials f for $\mathbf{F}$, i.e., all C^1 functions f such that $\nabla(f) = \mathbf{F}$.

1. $\mathbf{F} = (xz, xy, yz)$
2. $\mathbf{F} = (2y\sin(yz),\ 2x\sin(yz) + 3z + 2xyz\cos(yz),\ 2xy^2\cos(yz) + 3y)$

Problem 5: Let $a_{m,n} \geq 0$, and assume that each $a_{m+1,n} \leq a_{m,n}$ and $a_{m,n+1} \leq a_{m,n}$. Show that $\lim_m \lim_n a_{m,n} = \lim_n \lim_m a_{m,n} = a$, for some $a \geq 0$.

Problem 6: Let M be a compact metric space and $f : M \to M$ be continuous.

1. Let $M \times M$ be given metric $\rho((x, y), (x', y')) = d(x, x') + d(y, y')$. Show that the function $d(x, y)$ from $M \times M$ to $\mathbb{R}$ is continuous.

2. Let $r := \inf_{x \in M} d(x, f(x))$. Show that $r = d(x, f(x))$ for some $x \in M$.

3. Suppose that

$$\text{for all } x, y \in M \text{ s.t. } x \neq y,\ d(f(x), f(y)) < d(x, y). \tag{1}$$

   Show that f has a unique fixed point.