UBC Department of Mathematics Qualifying Exam in Analysis

January, 2017

Every problem is worth 10 points.

Problem 1: Let D_{2017} denote the dihedral group of the regular 2017-gon. Find the number of ordered commuting pairs of elements in D_{2017}, i.e. the size of

$$\{(a,b) \in D_{2017}^2 : ab = ba\}.$$

Problem 2: Let $f(x) = x^4 + x^2 + x + 1$ be a polynomial over the rational numbers.

1. Prove that $f(x)$ is irreducible over $\mathbb{Q}$.
2. Let s be a root of $f(x)$. Write down the following elements in the $\{1, s, s^2, s^3\}$ basis of $\mathbb{Q}(s)$:
   (a) s^7
   (b) s^{-1}.

Problem 3: Let $f(x) = x^4 - 2$ be a polynomial over the rational numbers and let E be the splitting field of $f(x)$ over $\mathbb{Q}$.

1. Prove that $E = \mathbb{Q}(\sqrt[4]{2}, i)$ and determine $\deg(E/\mathbb{Q})$.
2. Prove that $Gal(E/\mathbb{Q}) = D_4$.
3. Find every intermediate field K, between $\mathbb{Q}$ and E.
4. Show that the extension of $\mathbb{Q}$ by one root of $f(x)$ is not normal.

Problem 4: Find a matrix, $A \in \mathbb{R}^{2\times 2}$, satisfying

$$A = A^T, \quad A_{1,1} + A_{2,2} = 5, \quad \sum_{i,j} A_{i,j} = 19, \quad -A_{1,1} + A_{2,1} + A_{1,2} = 11.$$

Problem 5: Let $\mathcal{P}_2$ be the space of polynomials $a + bx + cx^2$ of degree at most 2 and with the inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x) \cdot q(x) dx.$$

1. Give an orthonormal basis for the orthogonal complement of $\operatorname{span}(x)$.
2. Let l be the functional defined by $l(p) := p(0)$ for each $p \in \mathcal{P}_2$. Find $h \in \mathcal{P}_2$ so that $l(p) = \langle h, p \rangle$ for each $p \in \mathcal{P}_2$.

Problem 6: Let $A, B \in \mathbb{R}^{3\times 3}$. Let $I \in \mathbb{R}^{3\times 3}$ be the 3 by 3 identity matrix. Suppose that A has eigenvalues $\{-1, 4, 10\}$ and B has eigenvalues $\{-2, 4, 7\}$. For each of the following matrices, if possible determine the eigenvalues. If not, state that there is insufficient information to determine the eigenvalues.

1. A^2.
2. $A \cdot B$.
3. $A + B$.
4. $A - 5 \cdot I$.
5. $A + A^{-1}$.