UBC Department of Mathematics Qualifying Exam in Differential Equations
September 6, 2016

Each problem is worth 10 points.

Problem 1: (a) Find the eigenvalues of the 3×3 matrix

$$A = \begin{pmatrix} 5 & -7 & 7 \\ 4 & -3 & 4 \\ 4 & -1 & 2 \end{pmatrix}.$$

(b) Is A diagonalizable?

Problem 2: Consider the ODE $y'' - y = e^{-t}$ for $y = y(t)$.

(a) Find the general solution.

(b) For which values of the initial conditions $y(0) = y_0$ and $y'(0) = v_0$, does the solution $y(t)$ remain bounded as $t \to \infty$?

Problem 3: Let

$$A = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \mathbf{0} \\ & & \ddots & \ddots & \\ & \mathbf{0} & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}$$

be an $n \times n$ Jordan block with eigenvalue λ.

(a) Suppose $|\lambda| < 1$. Show that $\lim_{d\to\infty} A^d = \mathbf{0}_{n\times n}$, where $\mathbf{0}_{n\times n}$ denotes the zero matrix of size $n \times n$.

(b) Suppose $\mathbf{w} := \lim_{d\to\infty} A^d \cdot \mathbf{v}$ exists and is non-zero for some vector $\mathbf{v}$ in $\mathbb{R}^n$. Show that this is only possible if $\lambda = 1$ and $\mathbf{w}$ is a non-zero scalar multiple of $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

Problem 4: Consider the following ODE system for $(x(t), y(t))$:

$$\begin{aligned} \tfrac{dx}{dt} &= y - y^4 \\ \tfrac{dy}{dt} &= x^2 - x \end{aligned}.$$

(a) Find all the critical points, and classify them as sinks, sources, saddles, or centres.

(b) Show that

$$H(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3 - \frac{1}{5}y^5$$

remains constant for solutions $(x(t), y(t))$.

(c) Prove that the origin $(x, y) = (0, 0)$ is *stable* in the sense that for any given $\epsilon > 0$, all solutions $(x(t), y(t))$ with initial conditions $(x(0), y(0))$ sufficiently close to $(0, 0)$, remain for all time within distance ϵ of $(0, 0)$.

Problem 5: In this problem n will denote a positive integer, A will denote an $n \times n$ matrix with real entries, and a_{ij} will denote the entry in the ith row and the jth column of A.

(a) Suppose $a_{ij} = x_i y_j$, where $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are n-tuples of real numbers. Show that $\text{rank}(A) \leq 1$.

(b) Let $f(x)$ be a polynomial of degree d with real coefficients, and $x_1, \ldots, x_n$ be real numbers. Consider the matrix A whose entries are given by the formula $a_{ij} := f(x_i + x_j)$. Show that $\text{rank}(A) \leq d + 1$.

Problem 6: Consider the following initial-boundary-value PDE problem for a function $u(x, t)$:

$$\begin{cases} u_t = u_{xx} + \alpha u^3 & 0 < x < 1, \ t > 0 \\ u(0, t) = u(1, t) = 0 \\ u(x, 0) = u_0(x) \end{cases} \tag{1}$$

where $\alpha \in \mathbb{R}$ is a parameter.

(a) If $\alpha = 0$ and $u_0(x) = x - x^2$, find the solution to (1).

(b) Show that if $u(x, t)$ is a smooth (infinitely differentiable function of (x, t)) solution of (1), then the quantity

$$E(t) = \int_0^1 \left(\frac{1}{2} (u_x(x, t))^2 - \frac{\alpha}{4} (u(x, t))^4 \right) \, dx$$

is non-increasing.

(c) For which values of α does problem (1) admit a non-zero static (time-independent) solution? Justify.