UBC Department of Mathematics Qualifying Exam in Algebra
September 6, 2016

Each problem is worth 10 points.

Problem 1: (a) Let $V \simeq \mathbb{R}^n$ be an n-dimensional real vector space and

$$f, g\colon V \to V$$

be linear transformations. Show that $f \circ g - g \circ f \neq \mathrm{id}_V$, where id_V denotes the identity transformation $V \to V$.

(b) Now suppose $V \simeq \mathbb{F}_p^n$ be the n-dimensional vector space over the field of p elements. Show that if $n = p$, then there exist linear transformations $f, g\colon \mathbb{V} \to V$ such that $f \circ g - g \circ f = \mathrm{id}_V$.

Problem 2: Does the alternating group A_4 have a subgroup of order 6?

Problem 3: Find the Jordan Canonical form of the matrix

$$A = \begin{pmatrix} 3 & -1 & 5 \\ 0 & 2 & 6 \\ 1 & -1 & 5 \end{pmatrix}$$

Problem 4: Let R be a commutative ring (with identity) and let I and J be distinct maximal ideals in R. Prove the following variants of the Chinese Remainder Theorem:

(a) The morphism $R \to R/I \times R/J$ given by $f\colon r \to (r \pmod{I}, r \pmod{J})$ is surjective.

(b) More generally, the morphism

$$R \to R/I^m \times R/J^n$$

given by $g\colon r \to (r \pmod{I^m}, r \pmod{J^n})$ is surjective, for any integers $m, n \geq 1$.

Problem 5: In this problem n will denote a positive integer, A will denote an $n \times n$ matrix with real entries, and a_{ij} will denote the entry in the ith row and the jth column of A.

(a) Suppose $a_{ij} = x_i y_j$, where $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are n-tuples of real numbers. Show that $\mathrm{rank}(A) \leq 1$.

(b) Let $f(x)$ be a polynomial of degree d with real coefficients, and $x_1, \ldots, x_n$ be real numbers. Consider the matrix A whose entries are given by the formula $a_{ij} := f(x_i + x_j)$. Show that $\mathrm{rank}(A) \leq d + 1$.

Problem 6: Let G be a finite group. Show that there exists a finite Galois field extension $K \subset L$ with $\mathrm{Gal}(L/K) = G$.