The University of British Columbia
Department of Mathematics
Qualifying Examination—Linear Algebra and Differential Equations
January 9, 2016

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1. Let J denote the $m \times m$ matrix of 1's.
   (a) (6 points) Show that J is a diagonalizable matrix. Give a basis for $\mathbf{R}^m$ consisting of eigenvectors for J.
   (b) (4 points) Show that if we have an $m \times n$ matrix A with $AA^T = 2J + 5I$ then $n \geq m$. (Fischer's inequality)

2. (a) (5 points) Given three mutually orthogonal vectors in $\mathbf{R}^3$, we can determine the matrices representing orthogonal projection onto each. What is the sum of the three matrices?
   (b) (5 points) We say $(a_1, a_2, a_3, \ldots)$ is a *fibonacci sequence* of real numbers if it satisfies the fibonacci recurrence namely if $a_{i+2} = a_{i+1} + a_i$ for $i = 1, 2, 3, \ldots$. Let U be the set of fibonacci sequences. Show that U is a vector space over $\mathbf{R}$ where we can define the addition of two sequences in the obvious way. Give the dimension of U.

3. Let A be an $n \times n$ matrix with real entries. Suppose $A^2 = -I$.
   (a) (2 points) Show that A is invertible (or *nonsingular*).
   (b) (2 points) Show that A has no real eigenvalues.
   (c) (3 points) Show that n must be even.
   (d) (3 points) Show that $\det(A) = 1$.

4. (10 points) Solve

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 3\cos 2t \\ 0 \end{bmatrix}$$

5. Use the method of Laplace transforms to solve the initial value problem

$$y'' + 5y = \begin{cases} t & \text{if } 0 \leq t < 1 \\ 2 & \text{if } t \geq 1 \end{cases}$$

   where $y(0) = 1$ and $y'(0) = 0$.

6. (10 points) Consider the differential equation

$$\frac{d^2y}{dx^2} + \sin y = 0$$

   (a) Convert this equation into a system of two first order differential equations.
   (b) Find all critical points (steady states, fixed points) of the system of part (a).
   (c) Classify the type of each of the critical points of part (b) and determine their stability.
   (d) Prove that if $\frac{dy}{dx}(0) = 0$ and $|y(0)| = a < \pi$, then $|y(x)| \leq a$ for all $x \geq 0$.
   (e) Prove that if $\frac{dy}{dx}(0) > 2$ and $y(0) = 0$, then $\lim_{x\to\infty} y(x) = +\infty$.

Table of Laplace transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = \text{positive integer}$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
$\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$t^n e^{at}, \quad n = \text{positive integer}$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$u_c(t) = \begin{cases} 0 & \text{if } x < c \\ 1 & \text{if } x > c \end{cases}$	$\frac{e^{-cs}}{s}, \quad s > 0$
$u_c(t) f(t-c)$	$e^{-cs} F(s)$
$u_c(t) g(t)$	$e^{-cs} \mathcal{L}\left\{g(t+c)\right\}(s)$
$e^{ct} f(t)$	$F(s-c)$
$f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right), \quad c > 0$
$\int_0^t f(t-\tau) g(\tau)\, d\tau$	$F(s) G(s)$
$\delta(t-c)$	e^{-cs}
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
$(-t)^n f(t)$	$F^{(n)}(s)$