The University of British Columbia
Department of Mathematics
Qualifying Examination—Analysis
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1. (a) Let $\mathcal{F}$ denote the family of all functions $\phi\colon \mathbb{R}^3 \to \mathbb{R}$ satisfying the identity

$$\nabla\phi(x,y,z) = \left(\ln(y^2+1) + z^2 e^{z^2 x}\right)\mathbf{i} + \frac{2xy}{1+y^2}\mathbf{j} + \left(2xze^{xz^2}+1\right)\mathbf{k}.$$

If $\mathcal{F}$ is empty, explain why; if $\mathcal{F}$ is not empty, identify all the functions it contains.

(b) For any smooth closed surface $\mathcal{S}$ in $\mathbb{R}^3$, oriented using outward normals, define the flux

$$\Phi[\mathcal{S}] = \iint_{\mathcal{S}} \mathbf{F} \bullet \widehat{\mathbf{n}}\, dS,$$

$$\text{where}\quad \mathbf{F} = \left(3y^2 - x^3 - x\cos z\right)\mathbf{i} - 4y^3\mathbf{j} + \left(3z + \sin z - xe^y - 9z^3\right)\mathbf{k}.$$

Identify the surface $\mathcal{S}$ that gives the largest possible value for $\Phi[\mathcal{S}]$.

2. Associate with every real-valued sequence $a_1, a_2, \ldots$, the values $\bar{a}, \bar{b} \in \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\bar{a} = \limsup_{n\to\infty} a_n, \qquad \bar{b} = \limsup_{n\to\infty} b_n, \quad \text{where} \quad b_n = \frac{a_1 + a_2 + \ldots + a_n}{n}.$$

(a) Present a sequence $\{a_n\}$ for which $\bar{a} \neq \bar{b}$.

(b) One of the inequalities $\bar{a} \le \bar{b}$ or $\bar{a} \ge \bar{b}$ holds in general. Decide which one, and prove it.

(c) Prove: If $|\bar{a}| < +\infty$ and $|a_n - \bar{a}| \to 0$ as $n \to \infty$, then $\bar{b} = \bar{a}$.

3. Here are three statements. For each statement, (i) define each underlined word or phrase, (ii) decide if the statement is true or false, and (iii) justify your decision by providing a suitable proof or counterexample.

(a) Whenever $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and $\underline{\{x_n\} \text{ is a Cauchy sequence in } \mathbb{R}}$, defining $y_n = f(x_n)$ makes $\{y_n\}$ a Cauchy sequence in $\mathbb{R}$.

(b) The only functions $f\colon [-1,1] \to \mathbb{R}$ that can be represented as a <u>uniform limit of polynomials</u> are the polynomial functions.

(c) The following function is <u>Riemann integrable on the interval $[0,1]$</u>:

$$f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{1}{n} \text{ for some } n = 1,2,3,\ldots, \\ x^2, & \text{otherwise.} \end{cases}$$

4. Determine the number of zeros of the polynomial $f(z) = z^4 - 5z + 1$ in the annulus $\{z :\ 1 < |z| < 2\}$.

5. Let U be the open upper half plane with the unit segment joining 0 to i deleted. In symbols,

$$U = \{z : \operatorname{Im} z > 0\} \setminus \{z : \operatorname{Re} z = 0 \text{ and } 0 < \operatorname{Im} z \le 1\}.$$

Find a conformal mapping of U onto the unit disk $\mathbb{D}(0,1) = \{z : |z| < 1\}$. (If your map involves a function defined using a branch cut, specify the precise branch you use.)

6. Suppose that f is analytic in the region $|z| < R$, except for a simple pole at z_0 with $0 < |z_0| < R$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the Taylor series of f at the origin. Show that the following limit exists and is not 0:

$$A = \lim_{n\to\infty} a_n z_0^{n+1}.$$