The University of British Columbia
Department of Mathematics
Qualifying Examination—Algebra
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1. (a) Find A^{10} for the matrix A given below:

$$A = \begin{bmatrix} -7 & -3 \\ 18 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}.$$

   (b) Let $P_2[x]$ denote the set of all polynomials in x having real coefficients and degree at most 2. Determine a basis for $P_2[x]$ which contains both $1 + x + x^2$ and $2 + x + x^2$.

   (c) Find all possible values of $\det(A + A^{-1})$, allowing arbitrary 3×3 matrices A with eigenvalues -1, 1, 2.

2. Let $M_{3\times3}$ be the vector space consisting of all 3×3 matrices (with elementwise addition and the usual scalar multiplication). Let 0 denote the zero matrix in $M_{3\times3}$. For each matrix A in $M_{3\times3}$, define

$$Z(A) = \{B \in M_{3\times3} \,:\, BA = 0\}.$$

   (a) Show that for each fixed $A \in M_{3\times3}$, the set $Z(A)$ is a subspace of $M_{3\times3}$.

   (b) Find the dimension of $Z(0)$, where 0 denotes the zero matrix in $M_{3\times3}$.

   (c) Suppose $A \in M_{3\times3}$ and $\text{rank}(A) = 2$. Find $\dim(Z(A))$.

3. Let k vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ in $\mathbb{R}^d$ be given. Assume that, for some constant $\alpha \in (0, 1)$,

$$\mathbf{u}_i^T \mathbf{u}_j = \begin{cases} 1, & \text{if } i = j, \\ \alpha, & \text{if } i \neq j. \end{cases}$$

   (Such a collection of unit vectors is called *equiangular*. Notice that the statement $\mathbf{u}_j^T \mathbf{u}_j = 1$ implicitly specifies that each $\mathbf{u}_j$ is a *column* vector of unit length.)

   Consider the set of $d \times d$ matrices $S = \left\{\mathbf{u}_i \mathbf{u}_i^T \,:\; i = 1, 2, \ldots, k\right\}$. Prove the following.

   (a) If the matrices in S are linearly independent, then $k \leq \dfrac{d(d+1)}{2}$.

   (b) The matrices in S are, in fact, linearly independent.
   Hint: One approach starts by postulating the matrix equation $\sum_{i=1}^k a_i \mathbf{u}_i \mathbf{u}_i^T = 0$, then multiplying from the left by $\mathbf{u}_j^T$ and from the right by $\mathbf{u}_j$.

4. Suppose $a, b \in \mathbb{C}$. Show that the ideal $(x - a, y - b)$ in $\mathbb{C}[x, y]$ generated by $x - a$ and $y - b$ is a maximal ideal.

5. Suppose p is a prime number. In parts (b)–(d) below, assume $d \geq 2$ is an integer. Recall that for any field $\mathbb{F}$, $GL_d(\mathbb{F})$ is the group of invertible $d \times d$ matrices with entries drawn from $\mathbb{F}$.

   (a) Show that $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$ is a field.

   (b) Show that $GL_d(\mathbb{Z}/p\mathbb{Z})$ is a finite group and compute its order in terms of d.

   (c) Let U be the subgroup of $GL_d(\mathbb{Z}/p\mathbb{Z})$ consisting of upper triangular matrices with all diagonal entries equal to 1. Find the order of U.

   (d) Show that $d!$ divides the order of $GL_d(\mathbb{Z}/p\mathbb{Z})$. Hint: realize the symmetric group as a subgroup.

6. (10 points) Part (b) below has many correct answers. Choose one that will help with part (c).

(a) Suppose $m, n \geq 1$ are integers. Compute the group of morphisms

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}).$$

(b) Find an example of a short exact sequence of abelian groups

$$0 \to G' \to G \to G'' \to 0.$$

(c) For the groups G, G', G'' in part (b), find an abelian group K for which the following sequence is NOT exact:

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(K, G') \to \mathrm{Hom}_{\mathbb{Z}}(K, G) \to \mathrm{Hom}_{\mathbb{Z}}(K, G'') \to 0.$$