# The University of British Columbia
## Department of Mathematics
## Qualifying Examination—Differential Equations
January 2025

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## Linear Algebra

1. 10 marks Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

   (a) Determine the eigenvalues of $A$. (Hint: 1 is an eigenvalue). [2 marks]

   (b) Determine the corresponding eigenvectors and a matrix $S$ that will diagonalize $A$ via a similarity transformation. [5 marks]

   (c) Does there exist a real $3 \times 3$ matrix such that $B^2 = A$? If so, explain how you would compute it, and if not, explain why not. [3 marks]

2. 10 marks (a) Let $u,\ v,\ w$ be three unit vectors in $\mathbb{R}^3$ such that

$$u \cdot v = -\frac{1}{2},\ u \cdot w = -\frac{1}{2},\ \text{and } v \cdot w = -\frac{1}{2}$$

   Show that $u + v + w = 0$. [2 marks]

   (b) Does there exist a real $2 \times 2$ matrix $A$ such that $A^3 = O$, but $A^2 \neq O$? ($O$ is the zero matrix). If so, give an example, and if not, explain why not.[3 marks]

   (c) Let $P$ be an orthogonal projection matrix, projecting onto the subspace $S$.

   - Let $R = I - 2P$. Show that $R$ is an orthogonal matrix and that $R^2 = I$. [3 marks]
   - Onto what space does $I - P$ project? [1 mark]
   - What is the relation between the nullspaces of $P$ and $I - P$? [1 mark]

3. 10 marks Consider the following discrete dynamical system representing the evolution of three distinct species:

$$\mathbf{u}_{n+1} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \mathbf{u}_n$$

(a) Show that the total population (sum of the three components of $\mathbf{u}$) is preserved from one generation $n$ to the next $n+1$. [2 marks]

(b) Determine the eigenvalues of the matrix. [3 marks]

(c) One of the eigenvalues is repeated. Find an orthonormal basis for the eigenspace associated with this eigenvalue. [3 marks]

(d) Starting with an initial population distribution $\mathbf{u}_0 = [9, 15, 6]^T$, determine the population distribution as $n \to \infty$. [2 marks]

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## Differential Equations

1. 10 marks Consider the conservative equation

$$x'' - x + x^3 = 0.$$

(a) Transform this 2nd-order ODE into a system of 1st-order ODEs by introducing a new variable $y(t) = x'(t)$. Find all fixed points (a.k.a. critical points or equilibria).

(b) Show that it is a conservative system by solving the quantity (also known as the energy) that is conserved by this system.

(c) For each fixed point found in (a), use linearization to determine the stability (if possible) and local behaviour (if possible using a local phase portrait in $x, y$-plane) near that point.

(d) Sketch the nullclines and the global phase portrait in $x, y$-plane. Combined with the conservation equation of this conservative system, filling up the parts of the phase portrait local phase portraits in (b) could not yield.

2. 10 marks Solve the following nonhomogeneous wave equation using standard separation-of-variables methods:

| | | | |
|---|---|---|---|
| (PDE) | $u_{tt} = u_{xx} + \pi^2 \sin \pi x,$ | $0 < x < 1,$ | $t > 0,$ |
| (BC) | $u(0,t) = 0,\ u_x(1,t) = -\pi,$ | | $t > 0,$ |
| (IC) | $u(x,0) = 2 \sin \pi x,$ | $0 < x < 1,$ | |
| | $u_t(x,0) = 0,$ | $0 < x < 1.$ | |

3. 10 marks Solve initial value problems using Laplace transformation. For a function $u(t)$, $t \geq 0$, define a linear operator $L$ such that $Lu = u'' + 3u' + 2u$.

(a) Solve the initial value problem $Ly = 0$, $y(0) = 1$, $y'(0) = -1$.

(b) Solve the initial value problem $Lv = \delta(t)$, $v(0) = 0$, $v'(0) = 0$.

(c) Solve the initial value problem $Lx = f(t)$, $x(0) = 1$, $x'(0) = -1$, and describe the behaviour of $x(t)$ for large $t$, where

$$f(t) = \begin{cases} 0, & \text{if } 0 \leq t < 3, \\ 4, & \text{if } t \geq 3 \end{cases} = 4h(t-3).$$

**TABLE 6.2.1** Elementary Laplace Transforms

| $f(t) = \mathcal{L}^{-1}\{F(s)\}$ | $F(s) = \mathcal{L}\{f(t)\}$ |
|---|---|
| 1. $1$ | $\frac{1}{s}$, $s > 0$ |
| 2. $e^{at}$ | $\frac{1}{s-a}$, $s > a$ |
| 3. $t^n$, $n =$ positive integer | $\frac{n!}{s^{n+1}}$, $s > 0$ |
| 4. $t^p$, $p > -1$ | $\frac{\Gamma(p+1)}{s^{p+1}}$, $s > 0$ |
| 5. $\sin at$ | $\frac{a}{s^2+a^2}$, $s > 0$ |
| 6. $\cos at$ | $\frac{s}{s^2+a^2}$, $s > 0$ |
| 7. $\sinh at$ | $\frac{a}{s^2-a^2}$, $s > \lvert a \rvert$ |
| 8. $\cosh at$ | $\frac{s}{s^2-a^2}$, $s > \lvert a \rvert$ |
| 9. $e^{at}\sin bt$ | $\frac{b}{(s-a)^2+b^2}$, $s > a$ |
| 10. $e^{at}\cos bt$ | $\frac{s-a}{(s-a)^2+b^2}$, $s > a$ |
| 11. $t^n e^{at}$, $n =$ positive integer | $\frac{n!}{(s-a)^{n+1}}$, $s > a$ |
| 12. $u_c(t)$ | $\frac{e^{-cs}}{s}$, $s > 0$ |
| 13. $u_c(t)f(t-c)$ | $e^{-cs}F(s)$ |
| 14. $e^{ct}f(t)$ | $F(s-c)$ |
| 15. $f(ct)$ | $\frac{1}{c}F\left(\frac{s}{c}\right)$, $c > 0$ |
| 16. $\int_0^t f(t-\tau)g(\tau)\,d\tau$ | $F(s)G(s)$ |
| 17. $\delta(t-c)$ | $e^{-cs}$ |
| 18. $f^{(n)}(t)$ | $s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}$ |
| 19. $(-t)^nf(t)$ | $F^{(n)}(s)$ |