The University of British Columbia
Department of Mathematics
Qualifying Examination—Differential Equations
September 2024

Linear Algebra

1. 10 marks Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

   (a) (4 points) Determine the eigenvalues of A. (Hint: -1 is an eigenvalue).

   (b) (3 points) Is A diagonalizable?

   - If A is diagonalizable, determine the corresponding eigenvectors and a matrix S that will diagonalize A via a similarity transformation.
   - If A is not diagonalizable, determine the generalized eigenvectors and a matrix Q that will transform A to Jordan canonical form via a similarity transformation. Write down a Jordan canonical form for this matrix.

   (c) (3 points) State and prove the Cayley-Hamilton theorem. (Hint: The Jordan canonical form could be useful).

2. 12 marks Let $W = span\{\vec{a}_1, \dots, \vec{a}_k\}$ be a $k-$dimensional subspace of $\mathbb{R}^n$ and A be the matrix whose columns comprise the vectors $\vec{a}_j$, $j = 1, \dots, k$.

   (a) (4 points) Let $\vec{b}$ be a vector in $\mathbb{R}^n$ and $\vec{p} \in W$ be the orthogonal projection of $\vec{b}$ onto W. Determine an expression in terms of A for the projection matrix P, which is such that $\vec{p} = P\vec{b}$.

   (b) (2 points) Show that P is idempotent and symmetric.

   (c) (2 points) Determine the eigenvalues of P and characterize its eigenvectors.

   (d) (2 points) Is the matrix representation of P unique? Motivate your answer rather than providing a detailed proof.

   (e) (2 points) Use the projection matrix P to write down the normal equation for the least squares solution to $Ax = b$.

3. 8 marks Consider the set of polynomials of degree n and the weighted inner product $\langle p, q \rangle = \int_{-1}^{1} \frac{p(x)q(x)dx}{\sqrt{1-x^2}}$.

You will find the following integrals useful:

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi, \quad \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}} dx = \frac{1}{2}\pi \quad \int_{-1}^{1} \frac{x^4}{\sqrt{1-x^2}} dx = \frac{3}{8}\pi$$

(a) (5 points) For $n = 2$ use Gram-Schmidt orthogonalization to determine polynomials $q_k(x)$ of degree k with $k = 0, 1, 2$ that are orthonormal with respect to the inner product $\langle ., . \rangle$.

(b) (3 points) How do the $q_k(x)$, $k \in \{0, 1, 2\}$ relate to the Chebyshev polynomials defined by $T_k(x) = \cos(k\theta)$, where $\theta = \arccos(x)$?

Differential Equations

1. 10 marks For the following linear system

$$\vec{x}'(t) = A\vec{x}(t), \qquad \text{where} \qquad A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}.$$

(a) Find any fundamental matrix, then calculate the matrix exponential e^{tA}.

(b) Use the result in (a) to find the general solution to the non-homogeneous system

$$\vec{x}'(t) = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

2. 10 marks Consider the conservative equation

$$x'' - x(x-1)(x-3) = 0$$

(a) Transform this 2nd-order ODE into a system of 1st-order ODEs by introducing a new variable $y(t) = x'(t)$. Find all fixed points (a.k.a. critical points or equilibria).

(b) For each fixed point found in (a), use linearization to determine the stability (if possible) and local behaviour (if possible using a local phase portrait in x, y-plane) near that point.

(c) Sketch the nullclines and the global phase portrait in x, y-plane. Combined with the conservation equation of this conservative system, filling up the parts of the phase portrait local phase portraits in (b) could not yield.

3. $\boxed{\text{10 marks}}$ Consider the following nonhomogeneous heat equation.

$$\begin{cases} u_t = u_{xx} - (u-1), \quad (0 < x < 1,\, t > 0), \\ u(0,t) = 1,\ u(1,t) = 2, \quad (t > 0), \\ u(x,0) = f(x), \quad (0 \le x \le 1). \end{cases}$$

(a) Solve for $u(x,t)$ for any smooth initial temperature distribution $f(x)$. Express $u(x,t)$ as an infinite series with coefficients expressed as an integral involving $f(x)$.

(b) For $f(x) = 1$, solve the coefficients of the series obtained in (a) explicitly.