The University of British Columbia
Department of Mathematics
Qualifying Examination—Analysis

January , 2025

Real analysis

1. 10 marks (a) State Stoke's theorem. You do not need to give it in full generality (the Calculus-4 version will be sufficient).

   (b) Now let R be a finite region in the xy plane, and let C be the boundary of R.

   - Suppose, further, that C consists of a single piecewise smooth closed curve that is oriented consistently with R.
   - Let $F(x, y), G(x, y)$ be real-valued functions with continuous partial derivatives everywhere on R.

   Use Stoke's Theorem to prove Green's Theorem in the plane:

   $$\oint_C (F(x,y)\mathrm{d}x + G(x,y)\mathrm{d}y) = \iint_R \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right) \mathrm{d}x\mathrm{d}y$$

   (c) Hence, evaluate

   $$I = \oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{r}$$

   where

   $$\mathbf{H} = \left(y^2 - e^{-y^2} + \sin(x), 2xye^{-y^2} + x\right)$$

   and C is the boundary of the triangle with vertices $(0, 0), (1, 0), (1, 2)$ oriented counter-clockwise.

2. 10 marks Let $f : (0, \infty) \to \mathbb{R}$ be twice differentiable. Further, suppose that there are constants A, B, C so that

   $$A = \sup \{|f(x)| \colon x > 0\} \qquad B = \sup \{|f'(x)| \colon x > 0\}$$
   $$C = \sup \{|f''(x)| \colon x > 0\}$$

   Prove that $B^2 \leq 4AC$ in the following three steps.

   (a) Use Taylor's Theorem to expand $f(x)$ about a and compute $f(a + 2h)$ for $h > 0$.

   (b) Rearrange the resulting expression to isolate $|f'(a)|$ and bound it in terms of A, C and h.

(c) Now pick h to minimise your bound on $|f'(a)|$ and clean up.

3. [10 marks] Let (X, ρ) and (Y, τ) be metric spaces, and let $f : X \to Y$ be a function.

   (a) Define what it means for f to be uniformly continuous.

   (b) Now let A be a non-empty subset of X, and

   - let f be a uniformly continuous function, and
   - let $(x_n)_n$ be a Cauchy sequence in A

   Prove that the sequence $\Big(f(x_n)\Big)_n$ is also Cauchy.

   (c) Construct an example that shows that if we weaken "uniform continuity" to "continuity", then the result is not true. That is, construct

   - two metric spaces,
   - a continuous function g that maps from one to the other, and
   - a Cauchy sequence $(y_n)_n$

   so that the sequence $\big(g(y_n)_n\big)$ is not Cauchy.

Complex analysis

To earn credits, you have to justify your answer carefully.

Notation:

- **Let** $D = \{z \in \mathbb{C} \mid |z| < 1\}$ **be the open unit disc in the complex plain** $\mathbb{C}$**.**
- $i = \sqrt{-1}$.

1. [10 marks] (a) [4 marks] Evaluate the contour integral

$$\int_C \frac{e^{z^2-2z+2}}{(z-1)^{100}} dz$$

where C is the circle in $\mathbb{C}$, of radius 10 centred at $z = 0$, with counter-clockwise orientation. Justify your answer.

(b) [6 marks] Compute the following integral for the real variable $x \in \mathbb{R}$ using **the contour integral method** in complex analysis:

$$\int_0^\infty \frac{\cos x}{(1+x^2)} dx.$$

This question asks you to use the contour integral method, so you must use it to earn a credit.

2. [12 marks] (a) [6 marks] Suppose two analytic functions $f, g : D \to \mathbb{C}$ satisfy $f(z_n) = g(z_n)$ for an infinite number of points $\{z_n\}_{n=1}^\infty \subset D$. Is it necessarily true that $f(z) = g(z)$ for all $z \in D$? Prove or give a counterexample; if you give a counterexample, you have to justify it.

(b) [6 marks] Let $f, g : \mathbb{C} \to \mathbb{C}$ be analytic functions. Suppose $|f(z)| > |g(z)|$ for all $z \in \mathbb{C}$; notice the strict inequality. Suppose that $g(0) = 0$. Find **all** $z \in \mathbb{C}$ such that $g(z) = 0$. Justify your answer.

3. [8 marks] Let $f, g : D \to \mathbb{C}$ be analytic functions. Suppose the following hold:

- $|f(z)| \geq |g(z)|$ for all $z \in D$; notice that the inequality is not strict;
- $f(0) = 0$ and $f'(0) = 1$;
- $g(0) = 0$, $g'(0) = 0$, and $g''(0) = 1$.

Can we determine the ratio $g(z)/f(z)$ as a function of $z \in D$ completely? If so give the formula for that ratio, otherwise give a counterexample. Justify your answer.