The University of British Columbia
Department of Mathematics
Qualifying Examination—Analysis
September, 2024

Real analysis

1. [10 marks] Let $\log x$ be the natural logarithm.
   (a) Prove that
   $$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$$
   (b) Prove that for $t > 0$
   $$\frac{t}{1+t} < \log(1+t) < t.$$
   (c) Finally, prove that the series
   $$\sum_{n\geq 0} |a_n|$$
   converges if and only if
   $$\sum_{n\geq 0} |\log(a_n+1)|$$
   converges.

2. [10 marks] Consider a function $f : [0,1] \to \mathbb{R}$.
   (a) Carefully define what it means for f to be Lipschitz continuous.
   (b) Prove that if f is Lipschitz continuous then it is uniformly continuous.
   (c) Prove that if f is uniformly continuous then it need not be Lipschitz continuous.

3. [10 marks] (a) Let (Y, τ) be a metric space and let $A \subseteq Y$. Define the function
   $$g : Y \to \{0,1\} \qquad g(y) = \begin{cases} 1 & y \in A \\ 0 & y \notin A \end{cases}$$
   Assume that the codomain of g lies in the usual metric space of the reals. Prove that g is discontinuous at all points z on the boundary of A.

(b) Let X, ρ be a metric space. Recall that a metric space is totally bounded when for any $\epsilon > 0$ there exist a finite collection of open balls of radius ϵ whose union contains X.

Prove that if every sequence in X has a Cauchy subsequence, then X is totally bounded.

Complex analysis

To earn credit, you have to justify your answers carefully.

1. [10 marks] (a) [5 marks] Let $g(z)$ be an analytic function on $\mathbb{C}$. Suppose that

$$g\left(\frac{1}{n}\right) = e^{-\frac{1}{n^2}} \qquad \text{for all natural numbers } n = 1, 2, 3, \cdots .$$

Compute $g^{(100)}(0)$, that is, the 100-th derivative of g at $z = 0$. Justify your answer.

(b) [5 marks] Find every continuously twice differentiable function $h(x, y)$ on $\mathbb{R}^2$ satisfying all of the following properties at the same time:

- $\dfrac{\partial^2 h}{\partial x^2}(x, y) + \dfrac{\partial^2 h}{\partial y^2}(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$, and
- $h(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2$, and
- $\dfrac{\partial h}{\partial x}(0, 0) = h(0, 0)$.

Justify your answer.

2. [10 marks] (a) [5 marks] Compute the following integral for the real variable $x \in \mathbb{R}$:

$$\int_0^\infty \frac{1}{(1 + x^2)^2} dx.$$

Justify your answer.

(b) [5 marks] Let $g(z), h(z)$ be analytic functions on the unit disk $D := \{z \in \mathbb{C} \mid |z| < 1\}$ such that h, g are inverse to each other, that is, $g(h(z)) = z = h(g(z))$ for all $z \in D$. Suppose $g(0) = 0$. Prove that the derivative $g'(z)$ is a constant function on D.

3. [10 marks] (a) [4 marks] Prove or disprove the following statement: There is an analytic function $f(z)$ on $A = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ such that

$$|z|^{-\frac{1}{3}} \leq |f(z)| \leq |z|^{-\frac{1}{2}} \text{ for all } z \in A.$$

(b) [6 marks] Find all analytic functions g on the unit disk $D := \{z \in \mathbb{C} \mid |z| < 1\}$ such that

$$|z|^{\frac{3}{2}} \leq |g(z)| \leq |z|^{\frac{1}{2}} \quad \text{for all } z \in D.$$

To earn credits, you have to justify your answer.