The University of British Columbia
Department of Mathematics
Qualifying Examination—Analysis
January, 2024

Real analysis

1. (10 points) Let C be the closed curve oriented counterclockwise consisting of the line segment from $(0,0)$ to $(1,0)$, the line segment from $(1,0)$ to $(1,1)$ and the part of the parabola $y = x^2$ from $(1,1)$ to $(0,0)$. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = xy\,\mathbf{i} + x^2\,\mathbf{j}$ by two methods:

   (a) (6 points) By calculating the line integral directly.

   (b) (4 points) By using Green's Theorem.

2. (10 points) (a) (5 points) Show that for $0 < x \leq y$ we have $\log(y) - \log(x) \leq y/x - 1$ where we are using the natural logarithm. (Hint: Write in terms of $z = y/x$ and apply the mean value theorem.)

(b) (5 points) Show that for all real numbers x we have $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \leq e^x$. For which x is it true that $1 + x + \frac{1}{2}x^2 \leq e^x$? Justify your answer.

3. (10 points) Let $E = \mathbb{Q} \cap [0, 1]$ be the set of rational numbers in the unit interval. For a function $f : E \to \mathbb{R}$ we say that f has a continuous extension g if (i) g is a real valued function defined for all real numbers in $[0, 1]$, (ii) $g(x) = f(x)$ whenever x is in E, (iii) g is continuous at every point in $[0, 1]$.

(a) (3 points) Suppose that f has two continuous extensions g and h. Prove that $g(x) = h(x)$ for $x \in [0, 1]$.

(b) (2 points) Define what it means for f to be uniformly continuous on E.

(c) (2 points) Suppose f has a continuous extension g. Show that this implies that f is uniformly continuous on E.

(d) (3 points) Give an example of a function f that is continuous on E and that has no continuous extension.

Complex analysis

4. (12 points) Compute the integrals

$$(a) \int_{|z|=1} e^{z+\frac{1}{z}} dz; \quad (b) \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx.$$

5. (8 points) Find the number of zeros of the polynomial $f(z) = z^4 - 6z + 3$ inside the annulus $1 < |z| < 2$.

6. (10 points) Find a conformal transformation that maps the region $\Omega = D \cap \{z \in \mathbb{C} : |z - 1| < 1\}$ onto

unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$.