The University of British Columbia
Department of Mathematics
Qualifying Examination—Algebra
January 2025

Linear Algebra

1. [10 marks] Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

   (a) Determine the eigenvalues of A. (Hint: 1 is an eigenvalue). [2 marks]

   (b) Determine the corresponding eigenvectors and a matrix S that will diagonalize A via a similarity transformation. [5 marks]

   (c) Does there exist a real 3×3 matrix such that $B^2 = A$? If so, explain how you would compute it, and if not, explain why not. [3 marks]

2. [10 marks] (a) Let $u,\ v,\ w$ be three unit vectors in $\mathbb{R}^3$ such that

$$u \cdot v = -\frac{1}{2},\ u \cdot w = -\frac{1}{2},\ \text{and } v \cdot w = -\frac{1}{2}$$

   Show that $u + v + w = 0$. [2 marks]

   (b) Does there exist a real 2×2 matrix A such that $A^3 = O$, but $A^2 \neq O$? (O is the zero matrix). If so, give an example, and if not, explain why not.[3 marks]

   (c) Let P be an orthogonal projection matrix, projecting onto the subspace S.

   - Let $R = I - 2P$. Show that R is an orthogonal matrix and that $R^2 = I$. [3 marks]
   - Onto what space does $I - P$ project? [1 mark]
   - What is the relation between the nullspaces of P and $I - P$? [1 mark]

3. [10 marks] Consider the following discrete dynamical system representing the evolution of three distinct species:

$$\mathbf{u}_{n+1} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \mathbf{u}_n$$

(a) Show that the total population (sum of the three components of $\mathbf{u}$) is preserved from one generation n to the next $n+1$. [2 marks]

(b) Determine the eigenvalues of the matrix. [3 marks]

(c) One of the eigenvalues is repeated. Find an orthonormal basis for the eigenspace associated with this eigenvalue. [3 marks]

(d) Starting with an initial population distribution $\mathbf{u}_0 = [9, 15, 6]^T$, determine the population distribution as $n \to \infty$. [2 marks]

Abstract Algebra

You can use any theorem from group theory, commutative algebra, Galois theory, etc. without proof as long as you state it clearly. The parts of all problems can be solved independently; if you do not know how to solve one of them, still try the later ones.

1. $\boxed{\text{10 marks}}$ Let a finite group G act on a finite set S. Denote by O_x the orbit and G_x the stabilizer of an element $x \in S$:

$$O_x = \{g \cdot x | g \in G\}, \quad G_x = \{g \in G | g \cdot x = x\}.$$

Let S^G denote the set of fixed points of the action:

$$S^G = \{x \in S | g \cdot x = x \text{ for all } g \in G\}.$$

Note that S^G consists of elements x whose orbit is the single element set $\{x\}$.

(a) (3 pts.) Prove that $|O_x| \cdot |G_x| = |G|$ for any $x \in S$. (Hint: consider the map $\phi : G \to S$, $\phi(g) = g \cdot x$. Study its image and its fibres.)

(b) (3 pts.) If $|G| = p^n$ for some prime p and integer $n > 0$, prove that

$$|S^G| \equiv |S| \pmod{p}.$$

(Hint: S is a disjoint union of orbits.)

(c) (4 pts.) If $|G| = p^n$ and $N \subseteq G$ is a normal subgroup, $N \neq \{1\}$, prove that

$$N \cap Z(G) \neq \{1\}.$$

Here $Z(G)$ is the centre of G:

$$Z(G) = \{g \in G | gh = hg \text{ for all } h \in G\}.$$

(Hint: let G act on N by conjugation and apply the previous part.)

2. $\boxed{\text{10 marks}}$ Consider field extensions $\mathbb{Q} \subseteq F \subseteq E$, where F is the splitting field of $x^3 - 3$ and E is the splitting field of $(x^3 - 3)(x^2 - 3)$. Let G be the Galois group of the extension $\mathbb{Q} \subseteq E$.

(a) (2 pts.) Show that E contains $i = \sqrt{-1}$.

(b) (3 pts.) Find the degrees of the extensions $\mathbb{Q} \subseteq F$, $F \subseteq E$, and $\mathbb{Q} \subseteq E$.

(c) (3 pts.) For each integer $n > 1$ dividing the order of G, determine if there exists a normal subgroup of G having order n. (Note that a subgroup of order n corresponds to a subfield $K \subseteq E$ such that $|E : K| = n$.)

(d) (2 pts.) For each prime p dividing $|G|$, find the number of p-Sylow subgroups of G and the same number of corresponding subfields of E.

3. $\boxed{\text{10 marks}}$ Let F be a field and let R be a commutative ring with 1.

(a) (2 pts.) Let $f, g \in F[x]$ be polynomials such that $\deg g \geq 1$. Prove that

$$f = f_0 + f_1 g + f_2 g^2 + \cdots + f_r g^r$$

for unique polynomials $f_i \in F[x]$ such that $\deg f_i < \deg g$ for all i.

(b) (2 pts.) Consider an infinite chain of prime ideals of R:

$$\ldots \subseteq P_{-2} \subseteq P_{-1} \subseteq P_0 \subseteq P_1 \subseteq P_2 \subseteq \ldots$$

Show that both $\cap_{i \in \mathbb{Z}} P_i$ and $\cup_{i \in \mathbb{Z}} P_i$ are prime ideals of R. You may assume without proof that they are both ideals.

(c) (3 pts.) Let P_1, P_2 be prime ideals of R and let $I \subseteq R$ be any ideal such that $I \nsubseteq P_j$ for $j = 1, 2$. Then there exists an element $r \in I$ such that $r \notin P_j$ for $j = 1, 2$.

(d) (3 pts.) Let P be a prime ideal of R and I_1, I_2 arbitrary ideals of R such that $I_j \nsubseteq P$ for $j = 1, 2$. Then there exists an element $r \in I_1 \cap I_2$ such that $r \notin P$.