The University of British Columbia
Department of Mathematics
Qualifying Examination—Algebra
September 2024

Linear Algebra

1. $\boxed{\text{10 marks}}$ Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

   (a) (4 points) Determine the eigenvalues of A. (Hint: -1 is an eigenvalue).

   (b) (3 points) Is A diagonalizable?

   - If A is diagonalizable, determine the corresponding eigenvectors and a matrix S that will diagonalize A via a similarity transformation.
   - If A is not diagonalizable, determine the generalized eigenvectors and a matrix Q that will transform A to Jordan canonical form via a similarity transformation. Write down a Jordan canonical form for this matrix.

   (c) (3 points) State and prove the Cayley-Hamilton theorem. (Hint: The Jordan canonical form could be useful).

2. $\boxed{\text{12 marks}}$ Let $W = span\{\vec{a}_1, \dots, \vec{a}_k\}$ be a $k-$dimensional subspace of $\mathbb{R}^n$ and A be the matrix whose columns comprise the vectors $\vec{a}_j$, $j = 1, \dots, k$.

   (a) (4 points) Let $\vec{b}$ be a vector in $\mathbb{R}^n$ and $\vec{p} \in W$ be the orthogonal projection of $\vec{b}$ onto W. Determine an expression in terms of A for the projection matrix P, which is such that $\vec{p} = P\vec{b}$.

   (b) (2 points) Show that P is idempotent and symmetric.

   (c) (2 points) Determine the eigenvalues of P and characterize its eigenvectors.

   (d) (2 points) Is the matrix representation of P unique? Motivate your answer rather than providing a detailed proof.

   (e) (2 points) Use the projection matrix P to write down the normal equation for the least squares solution to $Ax = b$.

3. $\boxed{\text{8 marks}}$ Consider the set of polynomials of degree n and the weighted inner product $\langle p, q \rangle = \int_{-1}^{1} \frac{p(x)q(x)dx}{\sqrt{1-x^2}}$.

You will find the following integrals useful:

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi, \quad \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}}dx = \frac{1}{2}\pi \quad \int_{-1}^{1} \frac{x^4}{\sqrt{1-x^2}}dx = \frac{3}{8}\pi$$

(a) (5 points) For $n = 2$ use Gram-Schmidt orthogonalization to determine polynomials $q_k(x)$ of degree k with $k = 0, 1, 2$ that are orthonormal with respect to the inner product $\langle ., . \rangle$.

(b) (3 points) How do the $q_k(x)$, $k \in \{0, 1, 2\}$ relate to the Chebyshev polynomials defined by $T_k(x) = \cos(k\theta)$, where $\theta = \arccos(x)$?

Abstract Algebra

You can use any theorem from group theory, commutative algebra, Galois theory, etc. without proof as long as you state it clearly. The parts of all problems can be solved independently; if you do not know how to solve one of them, still try the later ones.

1. $\boxed{\text{10 marks}}$ Let H be a subgroup of a finite group G. Then G acts on the set of left cosets of H by left multiplication:

$$g \cdot aH = gaH.$$

This defines a group homomorphism from G to the symmetric group of permutations of the set of left cosets of H:

$$\phi : G \to S_n.$$

(a) (2 pts.) Prove that $\ker \phi$ is a subgroup of H.

(b) (2 pts.) Find an example of H and G such that $\ker \phi$ is not equal to H.

(c) (3 pts.) If G has order 24, prove that G has a normal subgroup of order 4 or 8.

(d) (3 pts.) Let G have order 24 and assume that it has no normal subgroup of order 8. Then how many subgroups of order 8 does G have?

2. $\boxed{\text{10 marks}}$ Let E be the splitting field of the polynomial

$$f(x) = x^6 - 2 \in \mathbb{Q}[x].$$

(a) (2 pts.) Prove that $f(x)$ is irreducible in $\mathbb{Q}[x]$.

(b) (2 pts.) Find the degree of the extension $E/\mathbb{Q}$.

(c) (3 pts.) Let G be the Galois group of $E/\mathbb{Q}$. Note that $E = \mathbb{Q}(\xi, \sqrt[6]{2})$ is generated by a primitive 6-th root of unity ξ and a (real, positive) 6-th root of 2. Describe all elements of the Galois group by how they act on the two generators ξ and $\sqrt[6]{2}$. In particular, show that G can be generated by two elements τ and σ.

(d) (3 pts.) Find all intermediate fields $\mathbb{Q} \subseteq F \subseteq E$ such that $|F : \mathbb{Q}| = 3$. In particular, prove that the found intermediate fields are distinct and there are no more such fields.

3. $\boxed{\text{10 marks}}$ Let R be a commutative ring with $1 \neq 0$. Let $I, J, K \subseteq R$ be ideals, and let $P \subseteq R$ be a prime ideal. Prove:

(a) (2 pts.) If $IJ \subseteq P$ then $I \cap J \subseteq P$.

(b) (2 pts.) If $I \cap J \subseteq P$ then $I \subseteq P$ or $J \subseteq P$.

(c) (3 pts.) If $I \subseteq J \cup K$ then $I \subseteq J$ or $I \subseteq K$.

(d) (3 pts.) If $I \subseteq J \cup K \cup P$ then $I \subseteq J$ or $I \subseteq K$ or $I \subseteq P$. (Reduce to the previous part of the problem. To give a proof by contradiction, find $a, b, c \in I$ such that $a \notin J \cup K$, $b \notin J \cup P$, $c \notin K \cup P$, and study the element $a + bc$.)